A6. Calculating gravitational acceleration using a differential mathematical pendulum

***The aim of the exercise** is to understand the quantities describing the gravitational field, analyze harmonic motion using the example of a mathematical pendulum, and experimentally determine the acceleration due to gravity.*

The gravitational acceleration g is the acceleration imparted to a freely falling body. *The force of gravity F*, which is the force with which the Earth attracts the given body. We express it in m/s^2. If we denote the mass of the given body by *m*, the mass of the Earth by *M*, and its radius by *R*, then the formula for **gravitational force** is determined by the equation:

$$F = G\frac{M\,m}{R^2} \tag{1}$$

which is **Newton's law of gravity** (gravitational constant $G = 6.67 \cdot 10^{-11}\ Nm^2/kg^2$).

The mass of the Earth, like any mass, exerts an influence on the surrounding space, creating what is called **a gravitational field** around it. The intensity of this field is defined as $\gamma = F/m$, and on the surface of the Earth, it is numerically equal to the acceleration due to gravity $(\gamma = g)$.

A mathematical pendulum is a point mass (typically a small metal ball in practice) suspended on a weightless and inextensible thread. When the ball is displaced from its equilibrium position and released, it undergoes simple harmonic motion, driven by the tangential component of gravitational force (F). This force and the acceleration (a) it induces are proportional to the displacement (x) of the ball oscillating around its equilibrium position: $F = -m \cdot a = -m\omega^2 x$, where the angular frequency $\omega = 2\pi/T$, and *T* is the period of the pendulum's oscillations.

For small displacements, **the period *T* of the mathematical pendulum oscillations** is directly proportional to the square root of the pendulum's length (l), and inversely proportional to the square root of the acceleration due to gravity *g:*

$$T = 2\pi\sqrt{\frac{l}{g}} \tag{2}$$

The period *T*, however, does not depend on the amplitude of oscillations or the pendulum's mass.

Equation (2) can be used to calculate the acceleration due to gravity *g* if we measure the period *T* of the pendulum's oscillations and its length l. However, sometimes it is difficult to determine the length of the pendulum very precisely, i.e., the distance from the point of suspension to the center of mass of the ball. In such cases, it is useful to employ **a differential mathematical pendulum**, which is a regular mathematical pendulum modified to have a movable suspension point. By changing the slider's position on the scale, the pendulum's length is simultaneously changed. This method, therefore, does not require measuring the absolute length of the pendulum but allows for accurately determining changes in its length Δl. It is sufficient to measure the oscillation periods T_1 and T_2 for two different slider positions ((d_1 and d_2), corresponding to two different absolute lengths of the pendulum l_1 and l_2.

Using formula (2), we then obtain:

$$T_1 = 2\pi\sqrt{\frac{l_1}{g}} \qquad \text{and} \qquad T_2 = 2\pi\sqrt{\frac{l_2}{g}} \tag{3}$$

By squaring both sides of each of the above equations and subtracting them, we obtain the equation: $T_1^2 - T_2^2 = \frac{4\pi^2}{g}\Delta l$ (where: $\Delta l = l_1 - l_2 = d_1 - d_2$) which, after rearranging, yields the formula for the acceleration due to gravity:

$$g = 4\pi^2 \frac{\Delta l}{T_1^2 - T_2^2} \tag{4}$$

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Pair No.	Student's name and surname:	Field of study: Grupa:
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Measuring Protocol

Equipment: differential mathematical pendulum, period counter

Measuring steps:

1. Set the slider, adjusting the length of the mathematical pendulum, to the upper position of the ruler to obtain the maximum length of the pendulum. Read the position of the slider d_1.
2. Deflect the pendulum from the equilibrium position by a low angle ($\alpha \leq 4°$). Release the ball freely to maintain a single plane of oscillation.
3. Measure the time (t_1) using a counter for the duration of n complete oscillations.
4. Repeat the measurement three times.
5. Move the slider to shorten the pendulum length by 5-10 cm and read its position d_2.
6. Measure the time (t_2) for the duration of n oscillations of the pendulum with reduced length.
7. Following the procedure described in points 4-5, repeat the measurements for two other slider positions d_3 and d_4.

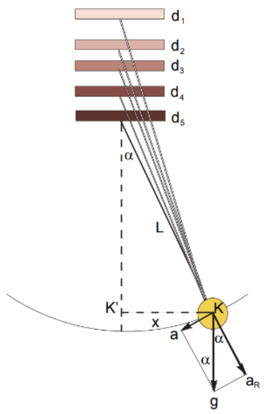

Figure.1. Differential mathematical pendulum.

Table 1 Number of pendulum oscillations n = ...

Slider position d [m]	d_1 =	d_2 =	d_3 =	d_4 =
Duration of n oscillations t [s]				
Average duration of n oscillations t_{avg} [s]	t_{avg1}=	t_{avg2}=	t_{avg3}=	t_{avg4}=
Period of oscillations T [s]	T_1=	T_2 =	T_3 =	T_4 =
T^2 [s^2]	T_1^2 =	T_2^2 =	T_3^2 =	T_4^2 =

Table 2

$\Delta l_{ij} = \lvert d_i - d_j \rvert$ [m]	$T_i^2 - T_j^2$ [s^2]	$g_{i,j}$ [m/s^2]	
$\lvert d_1 - d_2 \rvert =$	$T_1^2 - T_2^2$	$g_{12} =$	
$\lvert d_1 - d_3 \rvert =$	$T_1^2 - T_3^2$	$g_{13} =$	
$\lvert d_1 - d_4 \rvert =$	$T_1^2 - T_4^2$	$g_{14} =$	
$\lvert d_2 - d_3 \rvert =$	$T_2^2 - T_3^2$	$g_{23} =$	
$\lvert d_2 - d_4 \rvert =$	$T_2^2 - T_4^2$	$g_{24} =$	
$\lvert d_3 - d_4 \rvert =$	$T_3^2 - T_4^2$	$g_{34} =$	
	Mean value $\bar{g}$ =		Result statement ($\bar{g} \pm \Delta\bar{g}$) [$m/s^2$]
	Mean value error $\Delta\bar{g}$ =		

Data analysis and calculations

1. Based on the triple measurement of the duration of n pendulum oscillations performed for the slider position at d_1, calculate the mean time t_{avg1}, and then the period of oscillations ($T_1 = t_{avg1}/n$), and the square of the period of pendulum oscillations T_1^2 (Table 1).
2. Similar calculations are carried out for the measurements obtained for the pendulum of reduced length with the slider positioned successively at positions d_2, d_3, and d_4.
3. Calculate the differences in pendulum lengths ($\Delta l_{ij} = d_i - d_j$) [m] and the corresponding differences in the squares of the periods of oscillations ($T_i^2 - T_j^2$) [s^2] (Table 2).
4. Utilizing the formula:

$$g = 4\pi^2 \frac{\Delta l}{T_1^2 - T_2^2}$$

   and the data contained in Table 2, calculate the gravitational acceleration $g_{(ij)}$ based on all possible pairs of pendulum oscillation period measurements (T_i and T_j) obtained at different pendulum lengths (d_i and d_j) [m].
5. Calculate the average value of the gravitational acceleration $\bar{g}$ [m/s^2].
6. The absolute error is calculated using the standard deviation method:

$$\Delta\bar{g} = 3SD = 3\sqrt{\frac{\sum_{i=1}^{n}(\bar{g} - g_i)^2}{n(n-1)}}$$

7. Present the appropriately rounded average values of the gravitational acceleration $\bar{g}$ along with the error $\Delta\bar{g}$, in the form: ($\bar{g} \pm \Delta\bar{g}$) SI unit.