Fourth Moment of Random Determinant

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Abstract

Let X_{ij} be independent and identically distributed random variables, from which we construct matrices $A = (X_{ij})_{n\times n}$ and $U = (X_{ij})_{n\times p}$. We denote moments of their entries X_{ij} as $m_r = \mathbb{E}X_{ij}^r$ and their central moments as $\mu_r = \mathbb{E}\,(X_{ij} - m_1)^r$. Is there a way how we can express the even moments of determinants $\det A$ and $(\det U^\top U)^{1/2}$ in an exact form? That is, the objective is to find $f_k(n) = \mathbb{E}\,(\det A)^k$ and $f_k(n,p) = \mathbb{E}\,(\det U^\top U)^{k/2}$ as a function of m_r (or μ_r). Equivalently, one could first try to find the generating functions $F_k(t) = \sum_{n=0}^{\infty} \frac{t^n}{(n!)^2} f_k(n)$ and $F_k(t,\omega) = \sum_{n=0}^{\infty}\sum_{p=0}^{n} \frac{(n-p)!}{n!p!} t^p \omega^{n-p} f_k(n,p)$.

The exact expression for $F_2(t)$ and $F_2(t,\omega)$ can be easily derived using recurrences for any distribution of X_{ij}. For higher moments, it is not that simple. In the case of fourth moment, Nyquist, Rice and Riordan found the expression for $F_4(t)$ when $m_1 = 0$. Later, Dembo [2] derived $F_4(t,\omega)$ when $m_1 = 0$. The general case for both $F_4(t)$ and $F_4(t,\omega)$ when $m_1 \neq 0$ remained unsolved. However, as shown in recent arXiv preprint [1], we obtained

$$F_4(t) = \frac{e^{t(\mu_4 - 3\mu_2^2)}}{\left(1-\mu_2^2 t\right)^3}\left((1+m_1\mu_3 t)^4 + 6m_1^2\mu_2 t\,\frac{(1+m_1\mu_3 t)^2}{1-\mu_2^2 t} + m_1^4 t\,\frac{1+7\mu_2^2 t+4\mu_2^4 t^2}{\left(1-\mu_2^2 t\right)^2}\right),$$

$$F_4(t,\omega) = \frac{e^{t(\mu_4 - 3\mu_2^2)}}{\left(1-\mu_2^2 t\right)^2\left(1-\omega-\mu_2^2 t\right)}\Bigg[(1+m_1\mu_3 t)^4 + \frac{6m_1^2\mu_2 t(1+m_1\mu_3 t)^2}{1-\mu_2^2 t} + \frac{m_1^4 t\left(1+7\mu_2^2 t+4\mu_2^4 t^2\right)}{\left(1-\mu_2^2 t\right)^2}$$
$$+ \frac{\omega m_1^2 t}{1-\omega-\mu_2^2 t}\left(\frac{2\mu_2(1+m_1\mu_3 t)^2}{1-\mu_2^2 t} + \frac{m_1^2\left(1+5t\mu_2^2+2t^2\mu_2^4\right)}{\left(1-\mu_2^2 t\right)^2}\right) + \frac{2t^2\omega^2 m_1^4\mu_2^2}{(1-\omega-\mu_2^2 t)^2\left(1-\mu_2^2 t\right)^2}\Bigg].$$

One can easily deduce the moments $f_4(n)$ and $f_4(n,p)$ via Taylor expansion.

References

[1] D. Beck (2022). On the fourth moment of a random determinant. *Arxiv preprint arXiv:2207.09311 [math.pr].*

[2] A. Dembo (1989). On random determinants. *Quarterly of Applied Mathematics 47(2)*, 185–195.

[3] H. Nyquist, S. O. Rice and J. Riordan (1954). The distribution of random determinants. *Quarterly of Applied Mathematics 12(2)*, 97–104.