Some further applications of a lattice theoretic method in the study of singular LCM matrices

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Abstract

In 1876 H. J. S. Smith [5] defined an LCM matrix as follows: let $S = \{x_1, x_2, \ldots, x_n\}$ be a set of positive integers with $x_1 < x_2 < \cdots < x_n$. The LCM matrix $[S]$ on the set S is the $n \times n$ matrix with $\operatorname{lcm}(x_i, x_j)$ as its ij entry. During the last 30 years singularity of LCM matrices has interested many authors. In 1992 Bourque and Ligh [2] ended up conjecturing that if the GCD closedness of the set S (which means that $\gcd(x_i, x_j) \in S$ for all $i, j \in \{1, 2, \ldots, n\}$), suffices to guarantee the invertibility of the matrix $[S]$. However, a few years later this conjecture was proven false first by Haukkanen et al. [3] and then by Hong [4]. It turned out that the conjecture holds only on GCD closed sets with at most 7 elements but not in general for larger sets. However, the given counterexamples did not give much insight on why does the conjecture fail exactly in the case when $n = 8$. This situation was later improved in a couple of articles, where a new lattice theoretic approach was introduced (the method is based on the fact that because the set S is assumed to be GCD closed, the structure $(S, |)$ actually forms a meet semilattice). For example, it has been shown that in the case when the set S has 8 elements and the matrix $[S]$ is singular, there is only one option for the semilattice structure of $(S, |)$, namely the cube structure.

Since the cases $n \leq 8$ have been thoroughly studied in various articles, the next natural step is to apply the methods to the case $n = 9$. This was done by Altınışık and Altıntaş in [1] as they consider the different lattice structures of $(S, |)$ with nine elements that can result in a singular LCM matrix $[S]$. However, their investigation leaves two open questions, and the main purpose of this presentation is to provide solutions to them. We shall also give a new lattice theoretic proof for a result referred to as Sun's conjecture, which was originally proven by Hong via number theoretic approach.

Keywords

Bourque-Ligh conjecture, LCM matrix, GCD matrix, Smith determinant.

References

[1] Altınışık, E. and T. Altıntaş (2017). A note on the singularity of LCM matrices on GCD-closed sets with 9 elements. *Journal of Science and Arts 3(40)*, 413–422.

[2] Bourque, K. and S. Ligh (1992). On GCD and LCM matrices, *Linear Algebra Appl. 174*, 65–74.

[3] Haukkanen, P., J. Wang and J. Sillanpää (1997). On Smith's determinant, *Linear Algebra Appl. 258*, 251–269.

[4] Hong, S. (1999). On the Bourque-Ligh conjecture of least common multiple matrices, *J. Algebra 218*, 216–228.

[5] Smith, H. J. S. (1876). On the value of a certain arithmetical determinant, *Proc. London Math. Soc. 7*, 208–212.