



Shell-based numerical identification of effective transverse shear stiffness for thin-walled sections: isotropic steel benchmarks

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ABSTRACT

Effective transverse shear stiffness is a key parameter in reduced-order modelling of thin-walled members, particularly when complex cross-sectional geometries prevent reliable closed-form evaluation. This study develops a shell-based framework and compares three identification procedures using isotropic steel benchmarks: global compliance, volume-integrated transverse shear energy, and sectional shear fields. Eight open and closed cross-sections are examined in 114 numerical configurations, including variations of length, thickness, mesh size, sectional sampling position, end loading, and the local shell shear coefficient. Both transverse beam shear components are retained in the energy measures. Sectional estimates differ from elementary rectangular and circular thin-wall references by 0.040% and 0.024%, respectively, on the investigated 2 mm meshes. For the Sigma-section, refinement from 2 to 1 mm changes the sectional estimate by 0.136%, compared with 4.85% for global compliance identification. Longitudinal scans and alternative loading show that a stable central value does not by itself establish independence from the benchmark configuration. The framework distinguishes numerical consistency, spatial field development, and sensitivity to coupled deformation, providing qualified criteria for transferring shell-model information to reduced beam representations.

1. Introduction

Transverse shear deformation can contribute significantly to the response of beams with moderate or low slenderness ratios, thin-walled cross-sections, heterogeneous material layouts, or locally compliant structural components [1,2]. One-dimensional finite element formulations and classical plate models provide the broader computational background for the dimensional reduction of such structural members [3,4]. Euler–Bernoulli beam theory neglects transverse shear deformation and is therefore most appropriate when bending dominates the structural response. Timoshenko beam theory introduces an independent shear-deformation contribution through the effective transverse shear stiffness $(GA)_{\text{eff}}$, commonly expressed for an isotropic section as $(GA)_{\text{eff}} = k_s GA$, where G is the shear modulus, A is the selected reference area, and k_s is the shear correction factor [5–8]. The factor compensates for the difference between the simplified shear-strain field assumed in a one-dimensional beam model and the nonuniform shear field developing over the actual cross-section.

The interpretation of k_s is straightforward only when the reference

area, loading direction, beam kinematics, and contributing deformation modes are clearly defined. It is not a material constant and should not be expected to assume a universal value for geometrically different cross-sections. This distinction becomes particularly important for thin-walled open and asymmetric sections. Under transverse loading, such sections may exhibit coupled bending, torsion, nonuniform warping, and local cross-sectional distortion. A parameter identified solely from the global force–displacement response may then include contributions that cannot be attributed uniquely to transverse shear. In the present study, the term *shear correction factor* is therefore retained for cases in which the response can be interpreted predominantly through sectional shear deformation, whereas *effective transverse shear parameter* is used when the identified quantity remains dependent on the imposed benchmark configuration.

The classical foundations of shear correction were established by Timoshenko, who introduced approximate factors for simple solid cross-sections [9,10], and by Cowper, who derived Poisson-ratio-dependent expressions from complementary-energy considerations [11]. Refined elasticity and variational formulations subsequently improved the

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representation of nonuniform shear and warping fields [12–14]. These developments provide reliable analytical benchmarks for selected regular cross-sections, but closed-form solutions become increasingly difficult for thin-walled open profiles, multi-cell sections, curved walls, variable thicknesses, or geometries lacking double symmetry [15–18].

Computational dimensional reduction offers an alternative route. Detailed solid, shell, asymptotic, or high-order sectional models can be used to identify energetically equivalent stiffness properties that are subsequently transferred to one-dimensional beam formulations [19–25]. This does not imply that the reduced beam model is intended to reproduce every local feature of the detailed solution. Its purpose is to preserve the relevant global stiffness characteristics at a substantially lower computational cost. Such reduced representations are particularly valuable in large structural assemblies, repeated-load simulations, optimization, uncertainty analyses, and multiscale calculations.

Higher-order beam and plate theories likewise demonstrate the continuing need for mechanically consistent links between detailed cross-sectional behavior and reduced structural models [26,27]. Related numerical and structural formulations include enriched thin-walled beam elements, integral and mixed finite element approaches, specialized numerical quadrature, shell mechanics, variational asymptotic sectional analysis, and unified beam formulations [28–35].

The dimensional-reduction problem is directly relevant to composite structures, in which anisotropy, layered layouts, material interfaces, and coupling effects frequently require the identification of equivalent sectional stiffness matrices. Analytical extensions of shear coefficients to anisotropic beams and classical studies of Timoshenko-type correction further demonstrate that the identified factor depends on both the constitutive behavior and the kinematic assumptions of the reduced beam model [36,37]. The present investigation is intentionally restricted to isotropic, linear-elastic steel sections. Steel is used here as a controlled benchmark material so that the effects of geometry, boundary conditions, beam length, mesh size, and the numerical extraction procedure can be studied without simultaneously introducing constitutive anisotropy or interlaminar coupling. Extension to laminated or multi-material sections would require the corresponding anisotropic shell constitutive matrices and, in the general case, identification of a coupled sectional stiffness matrix rather than a single scalar shear factor. For open thin-walled sections, torsion and nonuniform warping may additionally require independent generalized measures or dedicated enriched finite elements [38].

Recent studies illustrate the breadth of coupled phenomena encountered in lightweight structural systems. Nonlinear dynamics and vibration in lightweight composite structures have been reviewed comprehensively in [39], while nonlinear forced vibrations of composite cylindrical shells under combined magneto-electro-thermo-mechanical loading provide an example of strongly coupled multiphysics behaviour [40]. Related nonlinear structural-dynamics research has also addressed vibro-impact energy-harvesting systems incorporating acoustic black holes [41]. Although these studies do not address shear-correction identification directly, they illustrate the broader need for reduced structural models whose effective parameters and retained kinematics remain compatible with the underlying deformation mechanisms.

Shell finite elements provide an efficient intermediate representation for this purpose. They retain the actual topology and plate-like mechanics of thin walls while avoiding the extensive through-thickness discretization required by three-dimensional solid models. Nevertheless, shell-based identification is not unique. Published procedures differ in the applied boundary conditions, load introduction, energy decomposition, selected strain components, sectional sampling, and normalization of the resulting stiffness [28–35]. Global compliance procedures are straightforward but may be affected by bending, torsion, warping, local distortion, and Saint-Venant end effects. Local energy and strain-based procedures provide more direct access to selected shear components, but their accuracy depends on strain recovery, mesh refinement,

and the position of the evaluated cross-section.

A broader historical synthesis of the development and interpretation of shear coefficients is provided by Elishakoff [42], while general theoretical foundations of plate and shell modelling are discussed by Altenbach [43]. Recent applications to layered corrugated board and porous eco-materials further show that analogous shear-stiffness identification problems arise in heterogeneous and multi-material systems [44–48].

Boundary conditions are especially important for open and asymmetric sections. Concentrated force application may introduce strong local stress disturbances, whereas displacement-controlled loading avoids selecting a single nodal load-application point and allows the resultant reaction force to be obtained from equilibrium. It does not, however, eliminate torsion, restrained warping, local distortion, or boundary-layer effects generated by the imposed kinematic constraints. Consequently, agreement between global and local identification procedures should be interpreted as a consistency indicator rather than assumed *a priori*.

Against this background, the present study develops a common shell-based finite element framework and compares three procedures for identifying effective transverse shear stiffness: (i) global compliance-based identification, (ii) volume integration of the complete transverse beam shear-energy contribution, and (iii) sectional shear-field identification.

All three procedures are applied to the same displacement-controlled models, material parameters, cross-sectional geometries, and discretization framework. The numerical investigation covers simple reference sections and more complex open and closed thin-walled profiles. The effects of member length, wall thickness, mesh size, and cross-sectional topology are examined. Verification is supported by comparison with classical analytical results for suitable simple sections and by a multi-level mesh-convergence study for a representative complex profile. Longitudinal scans, alternative end loading, shell-coefficient sensitivity, deformation fields, and sectional force-equilibrium checks extend this assessment.

The purpose of the study is not to introduce a new universal definition of k_s . Instead, it is to determine which numerical procedures provide stable and mechanically interpretable estimates under controlled conditions, to identify when a scalar beam-level parameter is sufficient, and to reveal when coupled deformation modes prevent its interpretation as an intrinsic sectional shear correction factor. Table 1 positions this contribution relative to the principal analytical and numerical approaches reported in the literature.

Recent developments further place this comparison within current sectional and shell modelling. Ding et al. [50] provide a generalized thin-walled sectional method, while Wu et al. [51] address dimensional reduction from elasticity. Polygonal Reissner–Mindlin formulations [52] and refined laminate shear coefficients [53] emphasize that element kinematics, recovery, and local constitutive correction are distinct modelling choices. The present contribution is a controlled finite-member comparison, not a replacement for established sectional solvers.

As summarized in Table 1, existing studies provide either analytical solutions for selected geometries, detailed high-fidelity reductions, or numerical estimates obtained using section-specific procedures. The present contribution differs by comparing three independent identification procedures within one controlled shell finite element environment. The same geometry, material model, loading mode, boundary conditions, and discretization strategy are retained for all procedures, allowing the influence of the extraction method to be distinguished from that of the underlying structural model. Particular attention is paid to mesh convergence, member-length sensitivity, and the distinction between sectional shear behavior and effective response contaminated by coupled torsion, warping, or local distortion.

Table 1

Representative approaches to the evaluation of transverse shear stiffness and their relation to the present study.

Representative approach	Section or material class	Main formulation	Principal output	Main limitation	Role in the present study
Timoshenko formulations [9,10]	Simple solid sections	Approximate shear-stress or shear-strain distributions	Classical shear correction factors	Restricted to regular geometries and simplified fields	Historical and analytical foundation
Cowper formulation [11]	Standard isotropic sections	Complementary-energy derivation including Poisson-ratio effects	Closed-form k_s expressions	Limited availability for complex thin-walled geometries	Analytical benchmark for suitable sections
Refined elasticity and variational formulations [12–14]	Selected solid and thin-walled sections	Higher-order fields, polynomial expansions, and energy principles	Refined shear factors and warping fields	Increasing analytical complexity	Theoretical reference
Thin-walled and warping theories [15–18]	Open and closed thin-walled sections	Shear-flow, torsion, and warping formulations	Sectional shear and warping stiffnesses	Geometry-specific assumptions and difficult automation	Basis for interpreting coupled deformation modes
Solid FE and asymptotic reduction [19–25]	Arbitrary, complex, and composite sections	Full elasticity or asymptotic dimensional reduction	Equivalent sectional stiffness matrix	High computational cost or specialized implementation	High-fidelity reference framework
Higher-order and unified beam formulations [26,27]	Homogeneous and composite members	Enriched beam kinematics	Reduced models with limited or no scalar shear correction	Increased number of generalized variables	Alternative reduced-order modelling strategy
Shell-based FE identification [28–35]	Thin-walled members	Section-specific shell modelling and numerical post-processing	Effective shear stiffness or k_s	Dependence on boundary conditions, mesh, and extraction method	Closest numerical family
Anisotropic shear, correction-factor, and warping formulations [36–38]	Anisotropic beams and open thin-walled sections	Constitutive extensions, correction-factor analysis, and torsion-warping finite elements	Effective shear, torsional, and warping parameters	Usually tailored to a selected constitutive or sectional class	Supports the interpretation of anisotropy and coupled deformation modes
Present study	Open and closed thin-walled sections, evaluated using isotropic steel benchmarks	Common shell FE framework with three independent identification procedures	Comparative $(GA)_{\text{eff}}$ and k_s assessment	Restricted to prismatic linear-elastic benchmark members	Controlled comparison, convergence assessment, and interpretation guidelines

2. Materials and methods

The numerical framework consists of four consecutive stages: construction of the shell finite element model from the midline representation of the thin-walled cross-section, recovery of the beam-oriented shear strain and energy measures from the shell solution, identification of the effective transverse shear stiffness using three complementary procedures, and assessment of the sensitivity of the identified parameters to geometry and numerical discretization. All procedures use the same material properties, shell formulation, loading mode, boundary conditions, and geometric models within each configuration. Consequently, differences between their predictions reflect the extraction procedure and its sensitivity to coupled deformation modes, boundary layers, discretization, and field recovery. The accompanying computational files contain the implementation, case definitions, raw fields, and numerical outputs.

2.1. Timoshenko beam background and definition of k_s

Consider a prismatic thin-walled member of length L , with its longitudinal axis aligned with the global x -direction and its cross-section located in the y - z plane. The prescribed transverse displacement acts in the global z -direction, and the corresponding bending deformation occurs primarily about the global y -axis.

The cross-sectional area A used throughout this study denotes the total material area of the thin-walled section, not the area of an arbitrarily selected web or another effective subregion. For a section represented by its wall midline Γ , it is evaluated as

$$A = \int_{\Gamma} t(s) ds, \quad (1)$$

where $t(s)$ is the local wall thickness and s is the curvilinear coordinate along the section midline. For a piecewise-constant discretization, the same quantity is calculated as $A = \sum_j t_j l_j$, where t_j and l_j are the thickness and midline length of the j -th wall segment. The second moment of area about the global y -axis is evaluated consistently from the same thin-

wall representation as

$$I_y = \int_{\Gamma} t(s) [z(s) - z_c]^2 ds, \quad (2)$$

where z_c is the centroid coordinate of the cross-section. These geometric quantities are determined directly from the cross-sectional midline geometry. Straight-segment integrals are evaluated exactly. All dimensions are midline dimensions; corner radii, junction overlap volumes, and local thickness-cubed corrections to the sectional area moments are excluded. The circular contour is represented by an inscribed regular polygon, and its polygonal properties are used consistently in the identification.

In a one-dimensional Timoshenko beam representation, the effective transverse shear stiffness is written as

$$(GA)_{\text{eff}} = k_s GA. \quad (3)$$

where G is the isotropic shear modulus and k_s is the beam-level shear correction factor. For the isotropic material considered in the present benchmark study, $G = E/(2(1 + \nu))$. Eq. (3) defines k_s through the selected reference area A . Consequently, a relatively low value of k_s does not necessarily indicate a low local shear stiffness. For a web-dominated section, for example, a considerable portion of the total material area may contribute primarily to bending rather than to transverse shear transfer.

For a cantilever member subjected to a resultant transverse force V at its free end, the Euler-Bernoulli bending contribution to the tip displacement is

$$w_b = \frac{VL^3}{3EI_y}. \quad (4)$$

The corresponding Timoshenko shear contribution is

$$w_s = \frac{VL}{(GA)_{\text{eff}}} = \frac{VL}{k_s GA}. \quad (5)$$

The total beam-level tip displacement is therefore

$$w = w_b + w_s = \frac{VL^3}{3EI_y} + \frac{VL}{(GA)_{\text{eff}}}. \quad (6)$$

The bending strain energy associated with the same reference cantilever response is

$$U_b = \frac{V^2 L^3}{6EI_y}, \quad (7)$$

whereas the shear strain energy is

$$U_s = \frac{V^2 L}{2(GA)_{\text{eff}}} = \frac{V^2 L}{2k_s GA}. \quad (8)$$

For a linear system subjected to a single resultant force V , the total strain energy satisfies $U_{\text{tot}} = Vw/2 = U_b + U_s$. It follows that global identification based on total compliance and global identification based on total strain energy are algebraically equivalent for the present displacement-controlled cantilever problem. They are therefore treated as one global compliance-based procedure in the revised framework rather than as two independent methods.

Equations (1)–(6) provide the beam-level reference relations used in the subsequent identification procedures. Their applicability requires that the shell response can be represented adequately by a bending-plus-shear beam model. For open or asymmetric sections exhibiting appreciable torsion, restrained warping, or local distortion, the stiffness obtained from the global response is interpreted as an effective parameter of the prescribed benchmark test rather than as a unique intrinsic sectional constant.

2.2. Shell finite element model and recovery of beam-oriented shear quantities

The thin-walled member is generated by extruding the midline representation Γ of an open or closed cross-section along the global beam axis x over a length L . The cross-sectional midline may consist of straight and curved segments with piecewise-constant or locally varying wall thickness. The extrusion produces a shell mid-surface model that retains

the actual topology of the walls without requiring solid discretization through their thickness.

One end cross-section, denoted by Γ_c , is fully clamped. All translational and rotational degrees of freedom at this boundary are set to zero. At the opposite end, Γ_f , the same prescribed transverse displacement $u_z = w_0$ is imposed on all end-section nodes, while the remaining degrees of freedom are left unconstrained. A unit displacement amplitude is used, $w_0 = 1$ in the adopted length unit. Because the analyses are geometrically and materially linear, the identified stiffness ratios are independent of the selected displacement magnitude.

The resultant transverse reaction force is calculated as

$$V = \left| \sum_{i \in \Gamma_f} R_{z,i} \right|. \quad (9)$$

where $R_{z,i}$ is the reaction force conjugate to the prescribed displacement of node i . The displacement-controlled configuration avoids the localized stress concentration associated with applying a concentrated nodal force and does not require a priori selection of a force-application point relative to the shear center. However, the uniform end displacement constitutes a specific kinematic constraint and may restrain torsion, warping, or local cross-sectional distortion in the vicinity of Γ_f . These effects are assessed through member-length variation, longitudinal scans, direct field diagnostics, and comparison with distributed-force loading, in addition to comparison among the three identification procedures.

The geometric idealization, coordinate systems, boundary conditions, and displacement-controlled loading scheme are shown in Fig. 1.

The numerical model employs a self-contained, displacement-based four-node Mindlin–Reissner flat-facet shell implementation, supplied in MATLAB. No commercial finite element is called. Each flat-panel node has five physical degrees of freedom: three translations and two tangential rotations. At a fold or junction, three global rotation components are retained to connect noncoplanar panels; at coplanar nodes the unobservable drilling rotation is removed by an exact change of nodal basis. No artificial drilling stiffness is introduced. The formulation

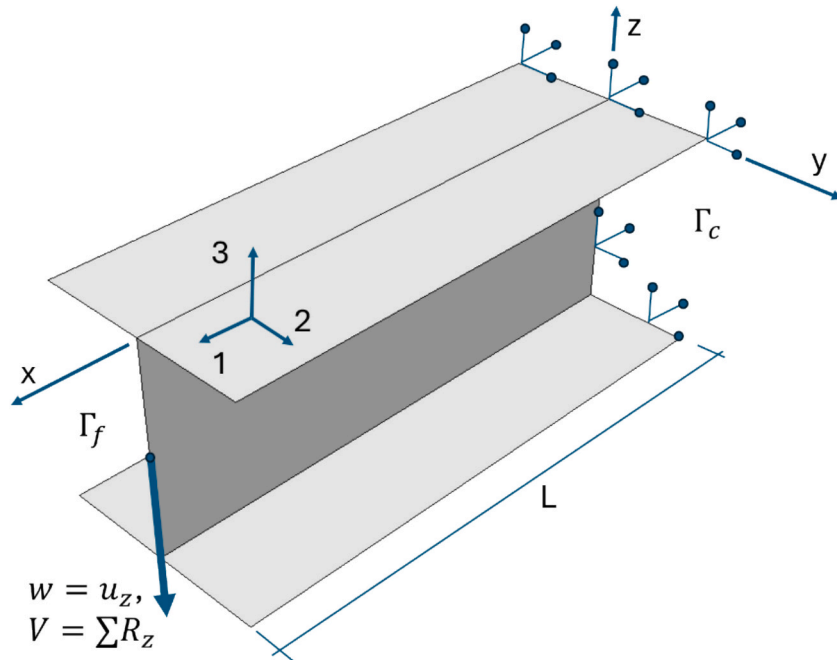


Fig. 1. Shell finite element model used for effective transverse shear-stiffness identification. The member of length L is aligned with the global x -axis, and the cross-section lies in the y - z plane. The end section Γ_c is fully clamped. A uniform prescribed transverse displacement $u_z = w_0$ is imposed on all nodes of the opposite end section Γ_f , and the resultant reaction force V is obtained by summing the corresponding nodal reactions. The inset presents the local shell coordinate system (1, 2, 3), where directions 1 and 2 span the shell mid-surface and direction 3 is normal to it.

includes neither thickness stretching nor mixed interpolation of tensorial components (MITC), assumed-strain stabilization, or reduced integration. The general finite element notation follows Hughes [49].

Full 2×2 Gauss integration is used for membrane, bending, and transverse-shear stiffness. Bilinear interpolation is applied to each physical nodal variable. Prescribed degrees of freedom are eliminated exactly, and the free system is solved directly after symmetric diagonal scaling. The standard shell-level transverse shear correction coefficient $\kappa_{sh} = 5/6$ is applied to the local transverse-shear constitutive terms associated with γ_{13} and γ_{23} . This shell-level coefficient accounts for the nonuniform through-thickness shear distribution within an individual homogeneous wall panel. It must be distinguished from the beam-level correction factor k_s , which characterizes the effective shear response of the complete thin-walled cross-section through Eq. (3). A complex assembly of homogeneous wall panels does not by itself invalidate this local assumption. Nevertheless, its effect on identification is tested explicitly by varying κ_{sh} from 2/3 to 1, corresponding to $\pm 20\%$ relative to 5/6; the results are reported in Section 3.9.4. The conclusion is not extended to laminated or anisotropic walls [53].

The present element is a conventional fully integrated displacement formulation and is therefore not claimed to be intrinsically free from transverse-shear locking in the thin limit. Its numerical behavior is examined using the multi-level mesh-convergence study and the wall-thickness sensitivity analysis described later. No general claim of locking-free or mesh-independent performance is made.

All benchmark sections are modelled as homogeneous isotropic linear-elastic steel with $E = 210$ GPa, $\nu = 0.30$, giving $G = 80.77$ GPa. The constitutive response is formulated under the plane-stress condition $\sigma_{33} = 0$, where direction 3 denotes the local shell-normal direction.

At each integration point, the shell generalized strain vector contains the in-plane and transverse-shear components

$$\boldsymbol{\epsilon}_{sh}^{(loc)} = [\epsilon_{11} \quad \epsilon_{22} \quad \gamma_{12} \quad \gamma_{13} \quad \gamma_{23}]^T. \quad (10)$$

For local variables $[u, v, w, r_1, r_2]$, the implemented membrane strains, curvatures, and transverse-shear strains are

$$\begin{aligned} \boldsymbol{\epsilon}_m &= [u_{,1}, v_{,2}, u_{,2} + v_{,1}]^T, \\ \boldsymbol{\chi} &= [r_{2,1}, -r_{1,2}, r_{2,2} - r_{1,1}]^T, \\ \boldsymbol{\gamma}_{sh} &= [w_{,1} + r_2, w_{,2} - r_1]^T. \end{aligned} \quad (11)$$

With the isotropic plane-stress matrix $\mathbf{D}$, the element stiffness is assembled as

$$\mathbf{K}_e = \int_{A_e} \left(\mathbf{t} \mathbf{B}_m^T \mathbf{D} \mathbf{B}_m + \frac{t^3}{12} \mathbf{B}_b^T \mathbf{D} \mathbf{B}_b + \kappa_{sh} G t \mathbf{B}_s^T \mathbf{B}_s \right) dA_e. \quad (12)$$

$$\mathbf{D} = \frac{E}{1 - \nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1 - \nu)/2 \end{bmatrix}. \quad (13)$$

For coordinate transformation, these components are embedded in the symmetric local strain tensor

$$\boldsymbol{\epsilon}^{(loc)} = \begin{bmatrix} \epsilon_{11} & \gamma_{12}/2 & \gamma_{13}/2 \\ \gamma_{12}/2 & \epsilon_{22} & \gamma_{23}/2 \\ \gamma_{13}/2 & \gamma_{23}/2 & 0 \end{bmatrix}. \quad (14)$$

The zero entry in Eq. (9) does not follow from the plane-stress condition. It represents the kinematic assumption of the adopted five-degree-of-freedom shell formulation, in which thickness-normal stretching is not included as an independent deformation mode. The plane-stress condition applies instead to the constitutive response and requires $\sigma_{33} = 0$. In a general three-dimensional plane-stress state, the physical thickness-normal strain need not be zero because of Poisson coupling.

The corresponding local stress components recovered from the shell constitutive law are embedded as

$$\boldsymbol{\sigma}^{(loc)} = \begin{bmatrix} \sigma_{11} & \tau_{12} & \tau_{13} \\ \tau_{12} & \sigma_{22} & \tau_{23} \\ \tau_{13} & \tau_{23} & 0 \end{bmatrix}, \quad (15)$$

where the transverse-shear stresses are evaluated using the shell-level correction coefficient:

$$\tau_{13} = \kappa_{sh} G \gamma_{13}, \tau_{23} = \kappa_{sh} G \gamma_{23}. \quad (16)$$

The in-plane shear stress τ_{12} follows the standard isotropic plane-stress constitutive relation and is not multiplied by κ_{sh} .

For each shell panel, the orthonormal transformation matrix

$$\mathbf{R} = [\mathbf{e}_1 \quad \mathbf{e}_2 \quad \mathbf{e}_3] \quad (17)$$

is constructed from the local unit vectors expressed in the global beam coordinate system. The local strain and stress tensors are transformed according to

$$\boldsymbol{\epsilon}^{(glob)} = \mathbf{R} \boldsymbol{\epsilon}^{(loc)} \mathbf{R}^T, \boldsymbol{\sigma}^{(glob)} = \mathbf{R} \boldsymbol{\sigma}^{(loc)} \mathbf{R}^T. \quad (18)$$

The beam-oriented transverse-shear strains and stresses are recovered in both transverse directions: $\gamma_{xa}^{beam} = 2\epsilon_{xa}^{glob}$, $\tau_{xa}^{beam} = \sigma_{xa}^{glob}$, $a \in \{y, z\}$. Both stress and strain are transformed before forming their work-conjugate product, rather than imposing the scalar relation $\tau_{xa} = G \gamma_{xa}$ after transformation. This retains the distinction between local in-plane shear and shell transverse shear, including the shell-level coefficient κ_{sh} . For the extruded facets, the local axes are $\mathbf{e}_1 = \mathbf{e}_x$, $\mathbf{e}_2 = \mathbf{t}$, and $\mathbf{e}_3 = \mathbf{e}_x \times \mathbf{t}$. Thus $\gamma_{xa} = t_a \gamma_{12} + n_a \gamma_{13}$ and $\tau_{xa} = G t_a \gamma_{12} + \kappa_{sh} G n_a \gamma_{13}$.

The complete transverse beam shear-energy density used in the two field-based procedures is consequently evaluated as

$$u_{\perp}^{beam} = \frac{1}{2} \left(\tau_{xy}^{beam} \gamma_{xy}^{beam} + \tau_{xz}^{beam} \gamma_{xz}^{beam} \right) = \frac{1}{2} G (\gamma_{12}^2 + \kappa_{sh} \gamma_{13}^2). \quad (19)$$

This quantity retains both transverse beam shear components. Using only the loading-direction product would omit energy carried in inclined and horizontal walls. It does not, however, provide a modal decomposition of translational shear, torsion, warping, or distortion: torsional shear may contribute to the same components. Local shell bending and twisting energies remain separate and are not silently assigned to this midsurface shear measure. The circular benchmark in Section 3.8 demonstrates the consequence of omitting the second transverse component.

2.3. Finite element strain energy and extracted response quantities

Let $\mathbf{q}$ denote the assembled vector of nodal displacements and rotations, and let $\mathbf{K}$ denote the global stiffness matrix of the shell finite element model. For the linear-elastic discretized system, the total strain energy is

$$U_{FE} = \frac{1}{2} \mathbf{q}^T \mathbf{K} \mathbf{q} = \frac{1}{2} \sum_e \int_{\Omega_e} \boldsymbol{\epsilon}_{sh}^T \boldsymbol{\sigma}_{sh} dV. \quad (20)$$

Here, ϵ_{sh} and σ_{sh} denote work-conjugate shell strain and stress fields, and Ω_e is the volume represented by element e . The total energy includes membrane, bending, and shell transverse-shear contributions. In the thickness integral, normal and in-plane shear strains include their curvature-dependent variation; they are not restricted to the midsurface values in Eq. (8). The separately integrated contributions are checked against the assembled quadratic energy.

Because the same displacement w_0 is prescribed at all nodes of the loaded end section, the total strain energy also satisfies $U_{FE} = V w_0 / 2$, where V is the resultant reaction force defined in Eq. (9). This equality provides a direct numerical check between the assembled finite element energy and the external work associated with the prescribed displacement.

The shell solution provides three groups of quantities used in iden-

tification: the global response pair (V, w_0) , the total strain energy U_{FE} , and both transverse beam shear stress and strain fields recovered from the shell solution. Method 1 uses global compliance, with total strain energy providing an algebraically equivalent formulation; Method 2 integrates the complete transverse beam shear-energy contribution over the model volume; and Method 3 evaluates the same work-conjugate components on a selected cross-section.

2.4. Identification of effective transverse shear stiffness

Three complementary procedures are used to identify effective transverse shear stiffness from the common shell solution. Method 1 is based on global compliance and is algebraically equivalent to a global energy expression for the linear displacement-controlled cantilever. Method 2 integrates the complete transverse beam shear-energy contribution over the shell model volume. Method 3 evaluates the work-conjugate transverse shear fields on a selected cross-section. Superscripts (1), (2), and (3) identify the procedures and do not denote exponentiation.

The procedures do not necessarily identify the same intrinsic sectional constant. Method 1 assigns to the residual response all deformation contributions not represented by the Euler–Bernoulli bending term. Methods 2 and 3 retain the global xy and xz shear stress–strain products but do not provide a complete decomposition of torsion, warping, or local distortion. Agreement is a useful consistency indicator, but cannot independently validate the common shell model. Disagreement is assessed against mesh sensitivity and direct field evidence before a physical interpretation is proposed.

2.4.1. Method 1: global compliance-based identification

The first procedure uses the resultant reaction force V and the prescribed displacement w_0 of the loaded end section. According to the Timoshenko cantilever relation in Eq. (4), the part of the global displacement attributed to transverse shear is

$$w_s^{(1)} = w_0 - \frac{VL^3}{3EI_y}. \quad (21)$$

The corresponding effective transverse shear stiffness is

$$(GA)_{\text{eff}}^{(1)} = \frac{VL}{w_s^{(1)}} = \frac{VL}{w_0 - \frac{VL^3}{3EI_y}}. \quad (22)$$

and the beam-level correction factor is

$$k_s^{(1)} = \frac{(GA)_{\text{eff}}^{(1)}}{GA}. \quad (23)$$

For the present linear displacement-controlled system, the same result can be expressed through global strain energy. Using $U_{FE} = Vw_0/2$, the effective shear-energy contribution is

$$U_s^{(1)} = U_{FE} - \frac{V^2L^3}{6EI_y}, \quad (24)$$

which gives

$$(GA)_{\text{eff}}^{(1)} = \frac{V^2L}{2U_s^{(1)}}. \quad (25)$$

The compliance and energy forms are algebraically identical and are therefore not treated as separate identification methods.

Method 1 is valid only when the denominator in Eq. (23) is positive and when the global response can be represented adequately by classical bending plus transverse shear. For open or asymmetric sections, torsion, restrained warping, local distortion, and end-constraint effects may contribute to w_0 . In such cases, $(GA)_{\text{eff}}^{(1)}$ is interpreted as an effective stiffness of the prescribed cantilever benchmark rather than as a unique sectional shear property.

The conditioning can be quantified without re-solving the shell problem. With the resultant and displacement fixed, define $\beta = w_b/w_0$. The local amplification due to uncertainty in the bending-reference stiffness is

$$\frac{\partial \ln k_s^{(1)}}{\partial \ln (EI_y)} = -\frac{\beta}{1-\beta}. \quad (26)$$

For a finite relative perturbation ε of EI_y , the exact ratio is

$$\frac{k_s^{(1)}(EI_y(1+\varepsilon))}{k_s^{(1)}(EI_y)} = \frac{1+\varepsilon}{1+\varepsilon/(1-\beta)}. \quad (27)$$

These expressions distinguish subtraction sensitivity from mechanical-model incompleteness. The $\pm 1\%$ examples in Section 3.9.5 perturb the postprocessing reference only; they are not physical material changes followed by a new shell solution.

2.4.2. Method 2: volume integration of the transverse beam shear energy

The second procedure uses the two transverse beam shear stress and strain fields recovered according to Section 2.2. Their combined energy is integrated over the complete shell representation of the member:

$$U_{\perp}^{(2)} = \sum_e \sum_{g \in \mathcal{G}} \frac{1}{2} \left(\tau_{xy}^{\text{beam}} \gamma_{xy}^{\text{beam}} + \tau_{xz}^{\text{beam}} \gamma_{xz}^{\text{beam}} \right)_{e,g} t_e \det(\mathbf{J}_{e,g}) w_g. \quad (28)$$

where e and g denote the element and integration point, respectively, t_e is the physical wall thickness, $\mathbf{J}_{e,g}$ is the in-plane isoparametric Jacobian, and w_g is the integration weight. Raw element Gauss-point fields are used for this volume integral.

The effective transverse shear stiffness follows from energetic equivalence with Eq. (8):

$$(GA)_{\text{eff}}^{(2)} = \frac{V^2L}{2U_{\perp}^{(2)}} \quad (29)$$

and

$$k_s^{(2)} = \frac{(GA)_{\text{eff}}^{(2)}}{GA}. \quad (30)$$

In contrast to Method 1, Method 2 does not subtract an Euler–Bernoulli bending contribution from the total shell energy. It directly integrates the two transverse beam shear stress–strain products. The method therefore avoids assignment of normal-strain bending energy to shear, but remains sensitive to the raw integration-point fields, mesh size, and boundary layers near both member ends.

The quantity $U_{\perp}^{(2)}$ is complete with respect to the two transverse beam shear components used here, but not with respect to every shell energy mechanism. It excludes normal-strain energy and local shell bending or twisting terms, while shear generated by torsion may remain included. This distinction is essential for open and asymmetric sections.

2.4.3. Method 3: sectional shear-field identification

The third procedure evaluates the beam-oriented shear response on a selected cross-section located at $x = x_s$. Unless stated otherwise, the reported sectional values are obtained at the member mid-length, $x_s = L/2$, which provides the greatest geometric separation from the two ends within a member of fixed length. This rationale does not establish that end effects are negligible. The revised calculations therefore scan 39 longitudinal positions, and representative length extensions are used to determine whether a stable central region develops.

Integration over the material area of the thin-walled cross-section is performed using its thickness-weighted midline representation:

$$\int_A f \, dA = \int_{\Gamma} f(s) t(s) \, ds. \quad (31)$$

The sectional resultant associated with the selected beam-oriented

shear stress is

$$V_{xz}(x_s) = \int_A \tau_{xz}^{\text{beam}}(x_s, y, z) dA, \quad (32)$$

and the corresponding complete transverse shear energy per unit member length is

$$U_{\perp}^{(3)}(x_s) = \frac{1}{2} \int_A (\tau_{xy}^{\text{beam}} \gamma_{xy}^{\text{beam}} + \tau_{xz}^{\text{beam}} \gamma_{xz}^{\text{beam}}) dA. \quad (33)$$

The effective sectional transverse shear stiffness is then

$$(GA)_{\text{eff}}^{(3)}(x_s) = \frac{V_{xz}(x_s)^2}{2U_{\perp}^{(3)}(x_s)}. \quad (34)$$

and

$$k_s^{(3)}(x_s) = \frac{(GA)_{\text{eff}}^{(3)}(x_s)}{GA}. \quad (35)$$

The area A in Eq. (35) is the total material area defined in Section 2.1. It is not restricted to the web and is not an independently prescribed effective shear area.

For each wall element, the strain–displacement operators are evaluated at the axial centre, $\xi = 0$, and at two transverse Gauss positions. Stress and strain are then interpolated separately in the physical axial coordinate between adjacent element centres before forming their energy product. The nearest centre field is used within an end half-element. No values are extrapolated to wall nodes and no averaging is performed across folds. The same rule applies at $x = L/2$ whether that point is inside an axial element or on an interface.

Sectional integration uses two Gauss points per wall segment. This centre-based recovery avoids arbitrarily selecting one of two derivative traces at an axial interface, but differs from M2 raw Gauss-point integration. Refinement and equilibrium checks therefore accompany the reported M3.

If both transverse shear pairs satisfy the pointwise isotropic constitutive relations $\tau_{xy}^{\text{beam}} = G\gamma_{xy}^{\text{beam}}$, $\tau_{xz}^{\text{beam}} = G\gamma_{xz}^{\text{beam}}$.

Eqs. (32)–(35) reduce to the corresponding strain-distribution form

$$k_s^{(3)} = \frac{(\int_A \gamma_{xz}^{\text{beam}} dA)^2}{A \int_A \left[(\gamma_{xy}^{\text{beam}})^2 + (\gamma_{xz}^{\text{beam}})^2 \right] dA}. \quad (36)$$

Equation (36) is a special isotropic reduction; it becomes the familiar single-component expression only if the second transverse component vanishes. The primary implementation uses the work-conjugate stress and strain fields in Eqs. (32)–(35), retaining the shell-level constitutive treatment in Section 2.2.

The signed sectional force discrepancy is defined as

$$e_V(x_s) = 100 \frac{V_{xz}(x_s) - V}{V}. \quad (37)$$

Method 3 provides a sectional estimate and avoids integrating boundary-layer contributions over the entire member length. Its reliability nevertheless depends on the selected longitudinal position. For very short members, the boundary layers generated at the two ends may overlap, and even the mid-length cross-section may not represent a Saint-Venant-type sectional state.

2.5. Numerical campaign, analytical verification, and robustness assessment

2.5.1. Parameter space

The numerical campaign includes eight representative thin-walled cross-sectional topologies: a vertical plate, an I-section, Sigma-, C-, Z-, and Omega-sections, a rectangular hollow section, and a circular hollow section. The sequence progresses from a simple analytical benchmark to

open asymmetric profiles and closed sections with increasingly complex shear-transfer paths.

Four member lengths are considered: $L = 50, 150, 250$, and 350 mm. The vertical plate and I-section are analyzed for wall thicknesses $t = 1, 2$, and 3 mm, whereas the remaining profiles are analyzed for $t = 1$ and 2 mm. The vertical-plate and I-section baseline campaigns use nominal in-plane element sizes of $h_e = 5$ mm and $h_e = 2$ mm. The remaining complex open and closed profiles are evaluated in the baseline campaign using $h_e = 5$ mm.

The Sigma-section is additionally subjected to a dedicated multi-level mesh-convergence study using $h_e = 5, 3, 2$, and 1 mm. The terms *coarse* and *fine* are used only for the two-level comparisons of the vertical plate and I-section and do not imply formal convergence.

For each cross-section, the characteristic depth H is the total midline extent in the global z -direction. Member slenderness is expressed by L/H . All eight benchmark geometries have $H = 100$ mm, so the baseline ratios are $L/H = 0.5, 1.5, 2.5$, and 3.5 .

The lowest ratios represent intentionally deep benchmark members and expose sensitivity to end constraints and non-beam deformation modes. They are not assumed to represent a Saint-Venant-type beam state. The additional Sigma, C, and circular models use $L = 700$ and 1400 mm to examine field development at $L/H = 7$ and 14 .

For every model, the extracted quantities include V , U_{FE} , the transverse beam shear fields, and $k_s^{(1)}$, $k_s^{(2)}$, and $k_s^{(3)}$. Mid-length M3 values are retained in the baseline tables, while the full longitudinal scans and equilibrium discrepancies are supplied separately. The baseline comprises 96 configurations; the additional mesh, length, loading, and shell-coefficient checks bring the total to 114 distinct solved models. Configurations shared by more than one diagnostic group are counted only once.

2.5.2. Analytical verification for the vertical-plate benchmark

The vertical-plate cross-section is geometrically equivalent to a narrow homogeneous rectangular section with depth H and width t . It is therefore used as an analytical benchmark for the implemented identification procedures.

In classical elementary shear theory, the Saint-Venant shear-stress distribution is evaluated as

$$\tau_{SV}(y, z) = \frac{VQ(y, z)}{I_y b(y, z)}, \quad (38)$$

where Q is the first moment of the portion of the cross-section above or below the evaluation point and b is the local section width. The corresponding energy-equivalent shear correction factor is

$$k_{s,SV} = \frac{V^2}{A \int_A [\tau_{SV}(y, z)]^2 dA}. \quad (39)$$

Because the factor V^2 appears in both the numerator and the squared stress field, Eq. (37) is independent of the selected load magnitude. For a homogeneous rectangular section, the classical parabolic shear-stress distribution gives

$$k_{s,SV} = \frac{5}{6} = 0.8333. \quad (40)$$

A second reference is provided by Cowper's Poisson-ratio-dependent expression for a rectangular isotropic section:

$$k_{s,C} = \frac{10(1 + \nu)}{12 + 11\nu}. \quad (41)$$

For the material parameter $\nu = 0.30$ used in the numerical models,

$$k_{s,C} = 0.8497. \quad (42)$$

The numerical values obtained for the vertical plate are compared with both analytical references. The value $5/6$ provides the classical energy-

equivalent benchmark based on the parabolic elementary stress distribution, whereas Cowper's expression additionally accounts for Poisson-ratio effects. The comparison is used to evaluate the implementation and should not be interpreted as requiring every numerical procedure to reproduce one of the two values exactly for very short, strongly constrained members.

2.5.3. Multi-level mesh convergence study

The two baseline element sizes, $h_e = 5$ mm and $h_e = 2$ mm, are insufficient by themselves to establish formal mesh convergence. Therefore, an additional multi-level convergence study is performed for the Sigma-section, selected as a representative complex open profile containing several wall junctions, free edges, and potential bending-torsion-warping coupling.

The convergence model uses $L = 350$ mm, $t = 2$ mm, and four nominal element sizes: $h_e = 5, 3, 2$ and 1 mm.

For each discretization, the total number of degrees of freedom and the three identified shear parameters are recorded. The relative deviation from the finest analyzed mesh is defined for Method m as

$$\eta_h^{(m)} = \frac{|k_s^{(m)}(h_e) - k_s^{(m)}(h_{\text{ref}})|}{|k_s^{(m)}(h_{\text{ref}})|} \times 100\%, h_{\text{ref}} = 1 \text{ mm}. \quad (43)$$

The 1 mm solution is treated as the finest available numerical reference rather than as an exact continuum solution. Practical convergence is considered achieved when the relative deviation from this reference is below 1%. If this criterion is not satisfied, the residual mesh sensitivity is reported explicitly, and no claim of mesh independence is made.

The primary convergence assessment concerns Methods 2 and 3 because they depend directly on recovered stress and strain fields. Method 1 is included for comparison, but its variation also reflects amplification in the bending subtraction. Results are shown versus element size and number of assembled physical degrees of freedom. Additional two-level checks for the C and circular sections test representative field-based results without being presented as formal convergence studies for every geometry.

2.5.4. Interpretation of member length and boundary effects

The member-length study serves two distinct purposes. First, it evaluates whether the identified quantity approaches a stable sectional value as the distance between the two constrained end regions increases. Second, it deliberately tests the breakdown of the one-dimensional bending-plus-shear interpretation for members with very low L/H .

For L/H close to unity or lower, the clamped-end boundary layer and the displacement-controlled end region may occupy a substantial part of the member and may overlap. Under these conditions, the response may contain restrained warping, torsion, local plate bending, and cross-sectional distortion in addition to the transverse shear deformation represented by a Timoshenko beam.

The three methods respond differently to these effects:

- Method 1 assigns to the residual compliance all deformation contributions not represented by the Euler–Bernoulli bending term.
- Method 2 retains both transverse beam shear stress–strain contributions, but integrates them over the full member volume and therefore includes both end regions.
- Method 3 evaluates the sectional shear field and is spatially the most selective procedure. Mid-length is the baseline location, but longitudinal scans are needed to establish whether it lies in a developed region.

Accordingly, substantial variation with L/H is treated as evidence of configuration dependence. A stable central region, low mesh sensitivity, and relevant length and boundary checks together provide stronger evidence for transferability. For open and asymmetric sections, method

disagreement alone is not used to identify torsion, warping, or distortion uniquely; deformation and stress fields provide supplementary evidence, with their diagnostic limitations stated explicitly.

2.5.5. Longitudinal scans and extended members

Sectional recovery is evaluated at 39 positions from $x/L = 0.025$ to 0.975 . A central-range indicator is the maximum minus minimum M3 over $0.30 \leq x/L \leq 0.70$, divided by the mid-length value. A range below 1% is a practical spatial-stability indicator, not proof of an intrinsic section property. Sigma, C, and circular members are also extended to $L = 700$ and 1400 mm at $t = 2$ mm and $h_e = 5$ mm.

2.5.6. Alternative end loading and local shell coefficient

For Sigma and C at $L = 350$ mm, $t = 2$ mm, and $h_e = 3$ mm, the baseline uniform displacement is compared with a uniform transverse line load along the end midline. Node weights are the sums of adjacent half-segment lengths, normalized to sum to one. All end degrees of freedom are free in this alternative, and the solution is scaled to a work-conjugate mean displacement of 1 mm:

$$f_{z,i} = Va_i, \sum_i a_i = 1, \bar{w} = \sum_i a_i u_{z,i} = 1 \text{ mm}, U_{FE} = \frac{1}{2} V \bar{w}. \quad (44)$$

This changes the end restraint and load distribution together; it is not an isolated release of warping or loading through the shear centre. For the same two profiles, $\kappa_{sh} = 2/3, 5/6$, and 1 are compared under the baseline end condition to quantify sensitivity to the local wall coefficient.

2.5.7. Equilibrium and deformation diagnostics

Free-equation residuals, assembled energy, external work, and the sum of membrane, bending, and shell-shear energies are checked. Eq. (37) is evaluated at every scan position. These algebraic checks do not constitute pointwise stress validation or exclude boundary effects.

Thickness-weighted least-squares fits remove rigid in-plane translation and rotation from sectional nodal displacements. The fitted rotation is reported as twist; the remaining root-mean-square displacement measures sectional distortion. Axial displacement is fitted to $a + b(y - y_c) + c(z - z_c)$, and the residual measures axial nonplanarity. Neither residual is presented as a unique classical warping mode or an orthogonal energy decomposition.

3. Results

The numerical results are presented in a sequence progressing from the elementary vertical-plate benchmark to increasingly complex open and closed thin-walled sections. For each geometry, the three identification procedures introduced in Section 2 are compared with respect to member slenderness, wall thickness, and baseline mesh size. The procedures are denoted by M1, M2, and M3, corresponding to global compliance-based identification, volume integration of the beam-oriented shear-energy contribution, and sectional shear-field identification, respectively.

The vertical plate is first used for analytical verification against the classical value $5/6$ and Cowper's Poisson-ratio-dependent solution. The subsequent profile-specific results examine the influence of cross-sectional topology and coupled deformation modes. A separate multi-level convergence study is presented for the Sigma-section to distinguish formal mesh refinement from the two-level baseline mesh comparison.

3.1. Vertical-plate benchmark and analytical verification

The first geometry considered is a single vertical thin-walled plate of height $H = 100$ mm, uniform thickness t , and member length L . The plate lies in the global $x - z$ plane and represents the simplest open thin-walled section transferring transverse shear in the investigated loading

direction.

This geometry is used as the primary analytical benchmark because its cross-section is equivalent to a narrow homogeneous rectangle. In a sufficiently developed beam state, the identified sectional shear correction factor can therefore be compared with the classical energy-equivalent value in Eq. (40) and with Cowper's value for $\nu = 0.30$, given in Eq. (42).

The geometry and an example of the shell finite element discretization are presented in Fig. 2.

The identified shear parameters and the corresponding reaction forces are summarized in Table 2. Because $H = 100$ mm, the four analyzed member lengths correspond to $L/H = 0.5, 1.5, 2.5$, and 3.5 .

To facilitate interpretation of the numerical data reported in Table 2, the corresponding variation of the identified shear parameters with L/H is shown in Fig. 3.

The reaction force increases linearly with wall thickness, whereas the normalized shear parameters remain unchanged within the reported precision. This behavior is consistent with linear elasticity: both the effective transverse shear stiffness and the reference stiffness GA scale linearly with t , so their ratio is independent of thickness for a geometrically similar homogeneous plate.

Method 3 provides the most stable estimate for the longer members. Its values range from 0.832 to 0.835 for $L/H \geq 1.5$, whereas the shortest member gives 0.870 and 0.873 on the 2 and 5 mm meshes. The latter values demonstrate that mid-length sampling does not remove overlapping end effects in a member with $L/H = 0.5$. At $L/H = 3.5$ and $h_e = 2$ mm, M3 is 0.834, differing from $5/6$ by 0.0395%. Cowper's 0.8497 remains a distinct reference based on different equivalence assumptions.

Method 2 exhibits a more pronounced dependence on member slenderness. For the baseline mesh with $h_e = 2$ mm, its value decreases from 0.885 at $L/H = 0.5$ to 0.824 at $L/H = 3.5$; for $h_e = 5$ mm, the corresponding variation is from 0.887 to 0.814. Because Method 2 integrates the selected shear-energy contribution over the complete member volume, its result includes the shear fields developing within both end regions and remains sensitive to integration-point field recovery.

Method 1 increases with L/H , from 0.882 to 0.924 for $h_e = 2$ mm and

from 0.887 to 0.949 for $h_e = 5$ mm. Two different effects must be distinguished. For the shortest member, the global response is affected by deep-member behaviour and end constraints. At larger L/H , the shear contribution becomes a small residual obtained after subtracting the dominant Euler–Bernoulli bending displacement from the total displacement. Consequently, even a small mismatch between the shell-model bending compliance and the reference EL_y term is amplified.

The two-level mesh comparison does not constitute a formal convergence study. For $L/H \geq 1.5$, the absolute M3 change between 2 and 5 mm meshes is below 0.0018; for $L/H = 0.5$, it is 0.0031. At $L/H = 3.5$, the corresponding changes are about 0.0092 for M2 and 0.0250 for M1. These results support sectional recovery as the primary plate verification measure, while quantifying its remaining discretization sensitivity. Formal multi-level refinement is assessed separately for Sigma.

Overall, the vertical-plate study verifies the sectional implementation against classical analytical solutions and demonstrates that agreement among the methods depends not only on cross-sectional geometry but also on member slenderness and on the spatial domain over which the response is evaluated. The benchmark therefore does not support interpreting every global or volume-integrated estimate as an intrinsic sectional constant.

3.2. I-Section: separation of shear-carrying and bending-dominated areas

The I-section is introduced as the first profile in which the regions contributing predominantly to bending and transverse shear can be distinguished clearly. The section has an overall height $H = 100$ mm, a flange width $B = 100$ mm, and a uniform wall thickness t . Under transverse loading in the global z -direction, the web provides the principal shear-transfer path, whereas the flanges contribute substantially to the total material area and bending stiffness.

This distinction is important for interpreting the beam-level correction factor. The reference area A in eq. (3) is the total material area of the web and both flanges, not the web area alone. For the nominal uniform-thickness midline dimensions used here, $A_w = Ht$, $A = t(H+2B)$, and therefore $A_w/A = H/(H+2B) = 1/3$ where A_w is the web area. This ratio is not an exact analytical shear correction factor, because it ne-

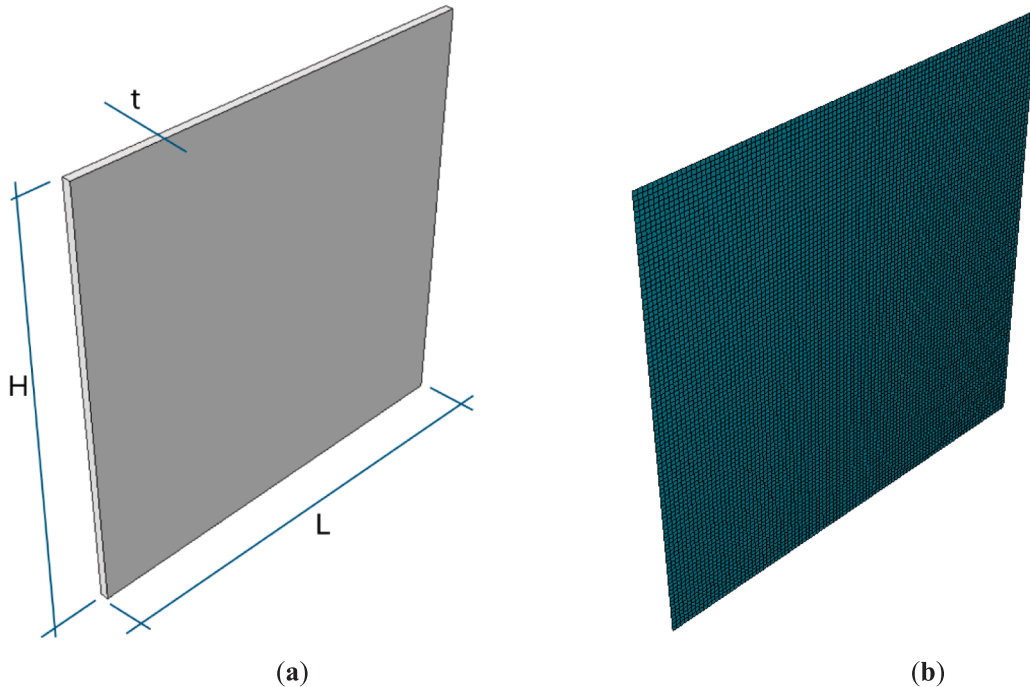
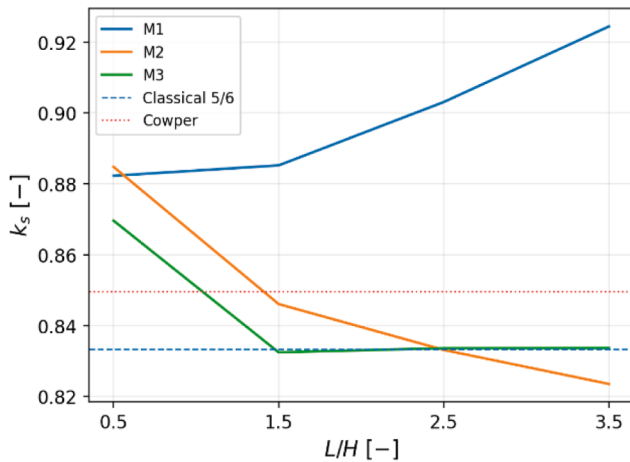


Fig. 2. Geometry and shell finite element model of the vertical-plate benchmark: (a) principal dimensions, with $H = 100$ mm, $t = 1, 2$, and 3 mm, and $L = 50, 150, 250$ and 350 mm; and (b) representative discretization with a nominal element size of $h_e = 2$ mm.

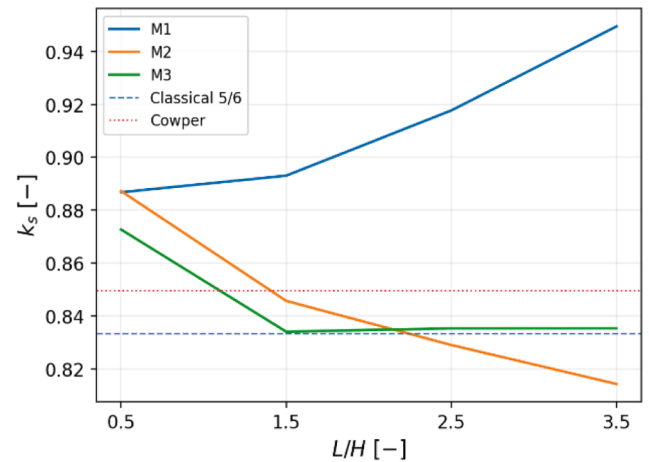
Table 2

Reaction force and effective shear parameters identified for the vertical-plate benchmark for different member slenderness ratios, wall thicknesses, and baseline mesh sizes.

L [mm]	L/H [-]	t [mm]	h_e [mm]	V [N]	$k_s^{(1)}$	$k_s^{(2)}$	$k_s^{(3)}$
50	0.5	1	2	106,412	0.882277	0.884794	0.869621
150	1.5	1	2	11,728	0.885211	0.846072	0.832450
250	2.5	1	2	3,013	0.903096	0.833134	0.833627
350	3.5	1	2	1,158	0.924416	0.823532	0.833663
50	0.5	2	2	212,824	0.882277	0.884794	0.869621
150	1.5	2	2	23,456	0.885211	0.846072	0.832450
250	2.5	2	2	6,026	0.903096	0.833134	0.833627
350	3.5	2	2	2,316	0.924416	0.823532	0.833663
50	0.5	3	2	319,236	0.882277	0.884794	0.869621
150	1.5	3	2	35,184	0.885211	0.846072	0.832450
250	2.5	3	2	9,039	0.903096	0.833134	0.833627
350	3.5	3	2	3,474	0.924416	0.823532	0.833663
50	0.5	1	5	106,818	0.886789	0.887238	0.872708
150	1.5	1	5	11,753	0.893052	0.845740	0.834097
250	2.5	1	5	3,018	0.917698	0.829073	0.835347
350	3.5	1	5	1,160	0.949429	0.814356	0.835381
50	0.5	2	5	213,636	0.886789	0.887238	0.872708
150	1.5	2	5	23,507	0.893052	0.845740	0.834097
250	2.5	2	5	6,036	0.917698	0.829073	0.835347
350	3.5	2	5	2,319	0.949429	0.814356	0.835381
50	0.5	3	5	320,454	0.886789	0.887238	0.872708
150	1.5	3	5	35,260	0.893052	0.845740	0.834097
250	2.5	3	5	9,054	0.917698	0.829073	0.835347
350	3.5	3	5	3,479	0.949429	0.814356	0.835381



(a)



(b)

Fig. 3. Effective shear parameters identified for the vertical-plate benchmark as functions of L/H : (a) $h_e = 2$ mm; and (b) $h_e = 5$ mm. The reference lines represent the elementary value 5/6 and Cowper's value 0.8497 at $\nu = 0.30$. The plotted values use $t = 1$ mm; normalized parameters are identical at the reported precision for the other investigated thicknesses. Curves have been regenerated from the supplied numerical outputs.

glects nonuniform shear distribution and the flange-web interaction. It nevertheless provides a useful geometric indicator: a value close to $1/3$ is physically plausible when transverse shear is carried predominantly by the web, but normalization is performed using the total section area.

The cross-sectional geometry and representative shell discretizations are presented in Fig. 4.

The four member lengths correspond to $L/H = 0.5, 1.5, 2.5$, and 3.5 . The identified parameters and reaction forces are summarised in Table 3.

The dependence of the three identified shear parameters on L/H , wall thickness, and baseline mesh size is illustrated in Fig. 5.

Method 3 remains below the geometric indicator $A_w/A = 1/3$. Across all I-section configurations it ranges from 0.296043 to 0.320785, approaching approximately 0.296 for the longest members. The difference from $1/3$ is consistent with nonuniform web shear and the com-

plete energy contribution of the flanges. Normalization by the total area remains essential: a value below that of the plate does not imply anomalously weak steel.

Method 2 is close to Method 3 for the longer members. On the 5 mm mesh, M2 ranges from 0.296455 to 0.327414; on the 2 mm mesh, it ranges from 0.296235 to 0.325240. At $L/H = 3.5$ and $t = 2$ mm, M2 and M3 are 0.296821 and 0.296289 on the baseline 5 mm mesh. The small separation must be assessed against recovery and mesh sensitivity before assigning it to end-region or junction effects.

Method 1 ranges from 0.299486 to 0.314530 over the I-section campaign. At $L/H = 3.5$ and $t = 2$ mm, it gives 0.302343 and 0.303442 on the 2 and 5 mm meshes, respectively. The results show only a weak variation with wall thickness. They still retain the inherent conditioning limitation of global subtraction, quantified independently in Section 3.9.5.

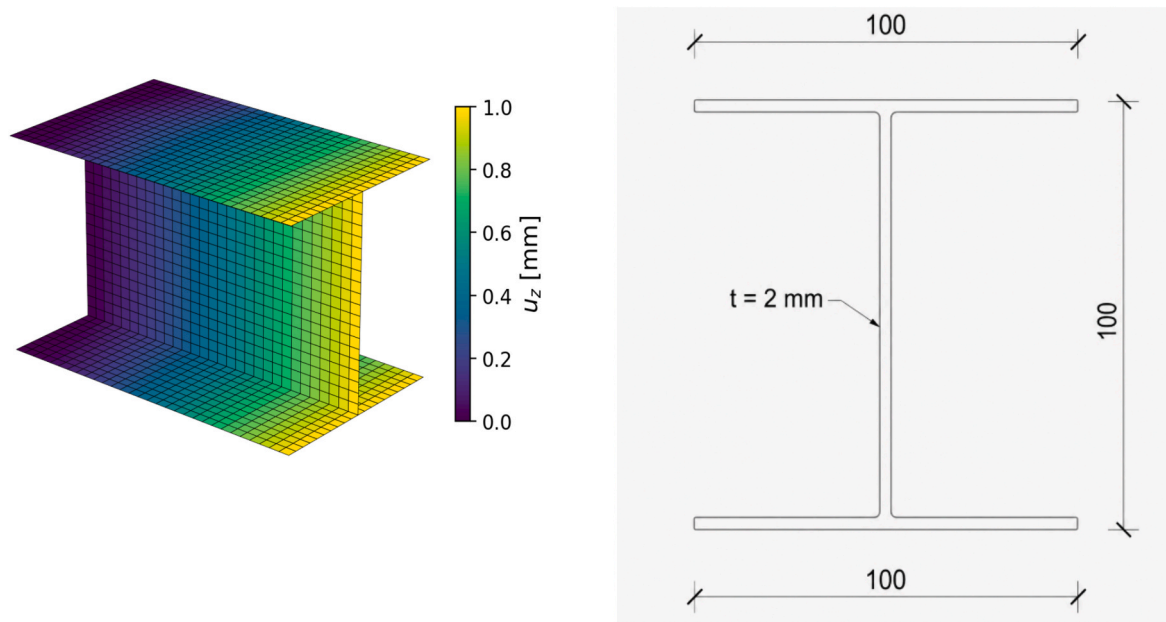


Fig. 4. Representative shell finite element model and cross-sectional geometry of the I-section: (a) executed model with $L = 150$ mm, $t = 2$ mm, and $h_e = 5$ mm, showing u_z on the undeformed mesh; (b) the original midline geometry and principal dimensions, with $H = B = 100$ mm and $t = 1, 2$, and 3 mm.

Table 3

Reaction force and effective shear parameters identified for the I-section for different member slenderness ratios, wall thicknesses, and baseline mesh sizes.

L [mm]	L/H [-]	t [mm]	h_e [mm]	V [N]	$k_s^{(1)}$ [-]	$k_s^{(2)}$ [-]	$k_s^{(3)}$ [-]
50	0.5	1	2	142,425	0.308854	0.319174	0.315256
150	1.5	1	2	33,496	0.299486	0.302238	0.298344
250	2.5	1	2	13,009	0.300333	0.298234	0.296495
350	3.5	1	2	6,079	0.301948	0.296235	0.296043
50	0.5	2	2	286,001	0.310167	0.321674	0.316276
150	1.5	2	2	67,073	0.300006	0.303076	0.298396
250	2.5	2	2	26,034	0.300745	0.298788	0.296539
350	3.5	2	2	12,162	0.302343	0.296674	0.296093
50	0.5	3	2	431,495	0.312062	0.325240	0.318029
150	1.5	3	2	100,780	0.300741	0.304231	0.298480
250	2.5	3	2	39,084	0.301325	0.299541	0.296608
350	3.5	3	2	18,253	0.302904	0.297261	0.296174
50	0.5	1	5	143,671	0.311697	0.321929	0.317800
150	1.5	1	5	33,593	0.300738	0.303143	0.298732
250	2.5	1	5	13,030	0.301433	0.298714	0.296662
350	3.5	1	5	6,086	0.303112	0.296455	0.296235
50	0.5	2	5	288,322	0.312814	0.324110	0.318887
150	1.5	2	5	67,253	0.301169	0.303852	0.298753
250	2.5	2	5	26,073	0.301773	0.299179	0.296714
350	3.5	2	5	12,175	0.303442	0.296821	0.296289
50	0.5	3	5	434,739	0.314530	0.327414	0.320785
150	1.5	3	5	101,031	0.301823	0.304904	0.298819
250	2.5	3	5	39,140	0.302290	0.299864	0.296787
350	3.5	3	5	18,272	0.303946	0.297354	0.296373

The two baseline meshes show relatively small differences for this profile, especially in M3 at the larger lengths. Nevertheless, two mesh levels do not establish a continuum limit, and agreement among procedures is not an independent validation because they share the same shell solution. Flange–web recovery, the fully integrated element, and end constraints remain possible sources of numerical or mechanical sensitivity.

The I-section illustrates why the total section area alone does not determine a shear factor. Its longer-member sectional values are stable and below the web-to-total-area indicator, while the global and integrated procedures give relatively close results in the audited campaign. It remains a useful intermediate case between the elementary plate and the more strongly coupled open profiles considered next.

3.3. Sigma-section: coupled open-section response and mesh convergence

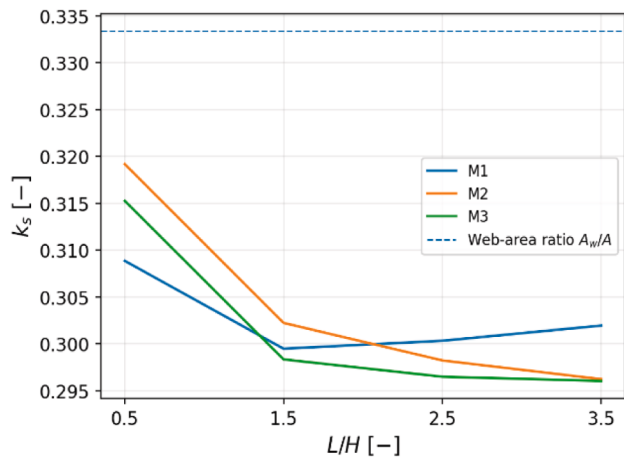
3.3.1. Baseline parametric results

The Sigma-section therefore serves two purposes in the present study. First, it is used to evaluate the behavior of the three identification procedures for a geometrically complex open section. Second, the configuration with $L = 350$ mm and $t = 2$ mm is selected for the dedicated multi-level mesh convergence study.

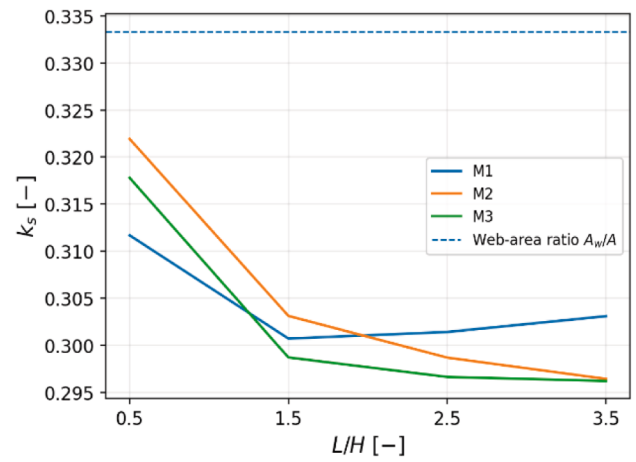
The cross-sectional geometry, representative shell model, and baseline results obtained using $h_e = 5$ mm are presented in Fig. 6.

The four member lengths correspond to $L/H = 0.5, 1.5, 2.5$, and 3.5 . The baseline results are reported in Table 4.

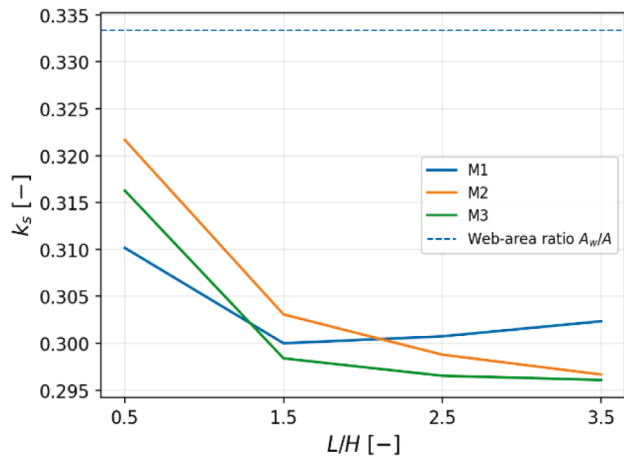
Method 3 decreases from 0.293991–0.295933 at $L/H = 0.5$ to 0.243966–0.244384 at $L/H = 3.5$. Thickness changes have a smaller



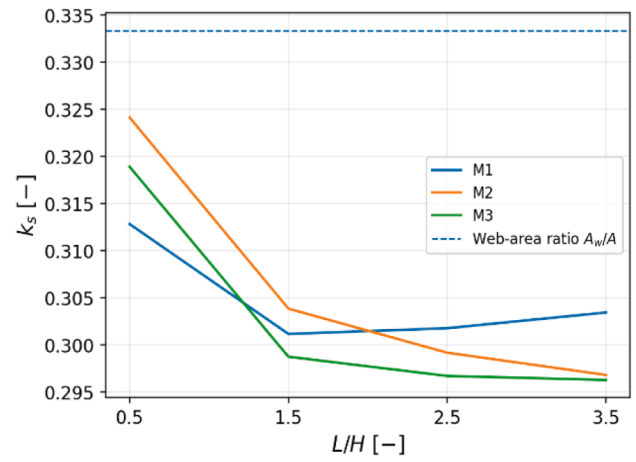
(a)



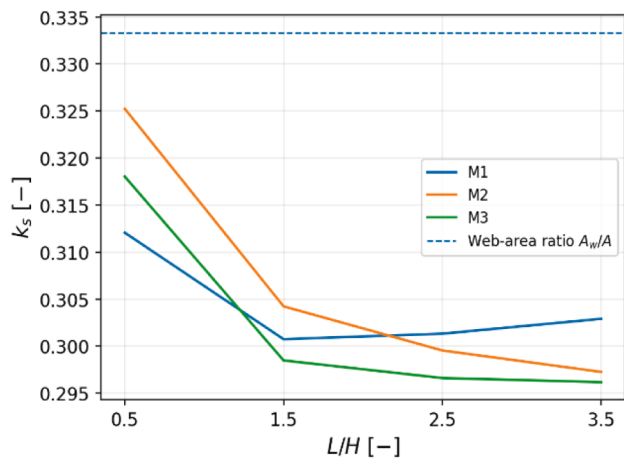
(b)



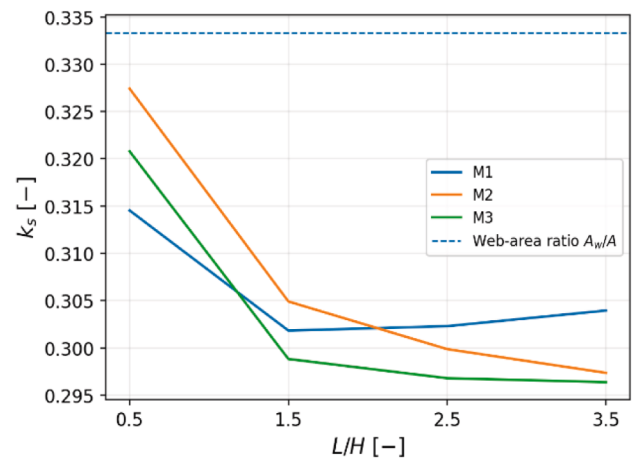
(c)



(d)



(e)



(f)

Fig. 5. Effective shear parameters identified for the I-section as functions of member slenderness L/H : (a) $t = 1$ mm, $h_e = 2$ mm; (b) $t = 1$ mm, $h_e = 5$ mm; (c) $t = 2$ mm, $h_e = 2$ mm; (d) $t = 2$ mm, $h_e = 5$ mm; (e) $t = 3$ mm, $h_e = 2$ mm; and (f) $t = 3$ mm, $h_e = 5$ mm. The horizontal reference line represents the first-order geometric ratio $A_w/A \approx 1/3$; it is not an exact analytical shear correction factor.

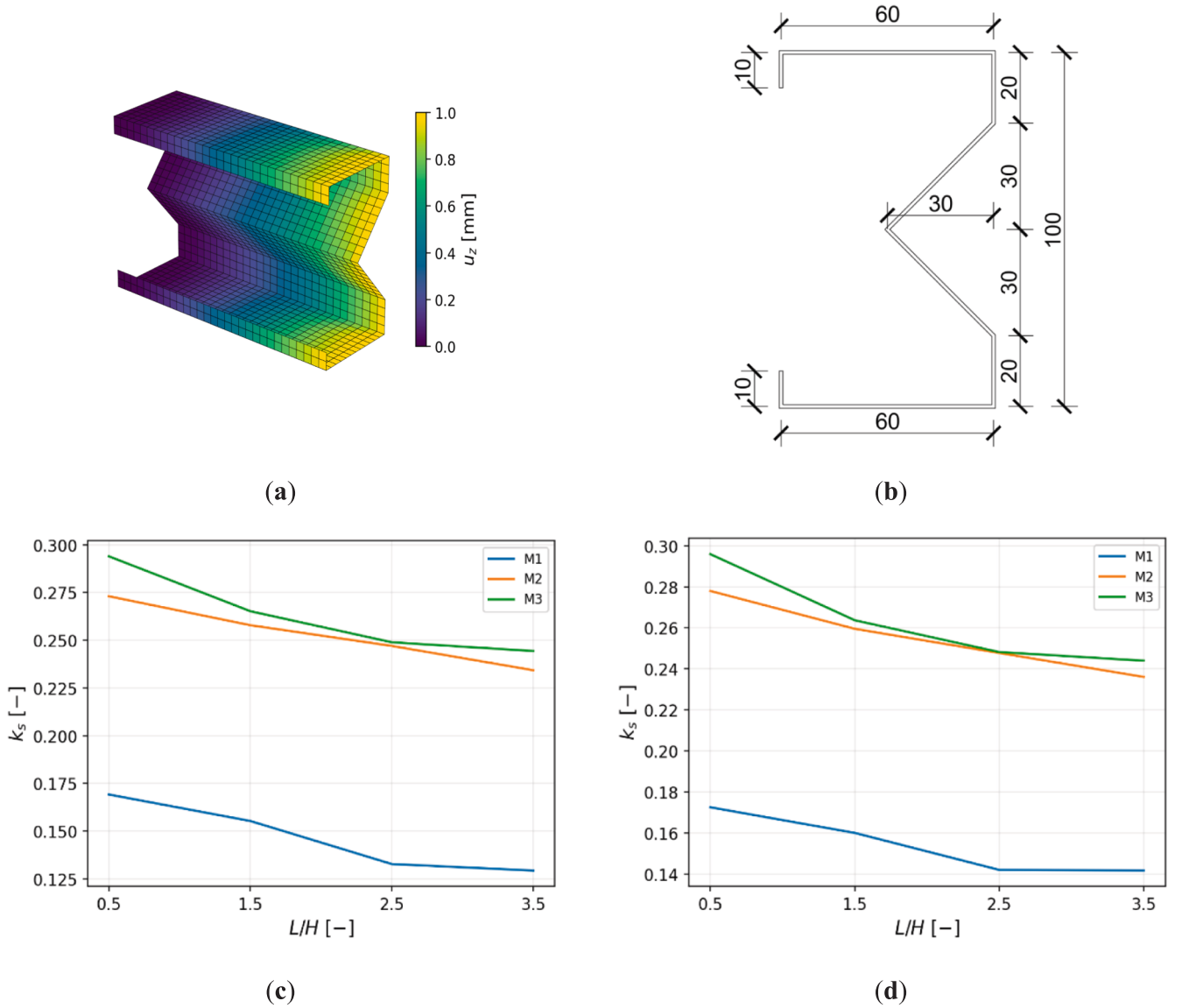


Fig. 6. Geometry, representative shell finite element model, and baseline effective shear parameters for the Sigma-section: (a) shell model with the transverse displacement field; (b) cross-sectional midline geometry and dimensions in millimeters, with an overall depth $H = 100$ mm; (c) results for $t = 1$ mm and $h_e = 5$ mm; and (d) results for $t = 2$ mm and $h_e = 5$ mm. The horizontal axes in panels (c) and (d) represent the member slenderness ratio L/H . Panel (a) shows the executed displacement field on the undeformed mesh for $L = 150$ mm, $t = 2$ mm, and $h_e = 5$ mm.

Table 4

Reaction force and effective shear parameters identified for the Sigma-section using the baseline mesh $h_e = 5$ mm.

L [mm]	L/H [-]	t [mm]	h_e [mm]	V [N]	$k_s^{(1)}$ [-]	$k_s^{(2)}$ [-]	$k_s^{(3)}$ [-]
50	0.5	1	5	70,071	0.169232	0.273064	0.293991
150	1.5	1	5	17,375	0.155344	0.257909	0.265230
250	2.5	1	5	6,873	0.132782	0.247057	0.248970
350	3.5	1	5	3,519	0.129394	0.234305	0.244384
50	0.5	2	5	142,763	0.172504	0.277975	0.295933
150	1.5	2	5	35,559	0.159989	0.259533	0.263694
250	2.5	2	5	14,305	0.141959	0.247738	0.248184
350	3.5	2	5	7,321	0.141699	0.236051	0.243966

effect than length in this baseline comparison, but the sectional response cannot yet be described as independent of the benchmark geometry. The longitudinal scans in Section 3.9.1 show that the central range still reaches 8.48% at $L/H = 3.5$.

Method 2 is below Method 3 in the regenerated baseline results. For $t = 1$ mm it decreases from 0.273064 to 0.234305; for $t = 2$ mm it de-

creases from 0.277975 to 0.236051 as L/H increases from 0.5 to 3.5. The integrated measure samples both end regions and uses raw Gauss-point fields, whereas M3 uses the specified sectional recovery. Their difference therefore combines spatial averaging and recovery effects.

Method 1 differs substantially from both field-based procedures. For $t = 1$ mm it decreases from 0.169232 to 0.129394, while for $t = 2$ mm it

changes from 0.172504 to 0.141699. At $L/H = 3.5$ and $t = 2$ mm, $M1 = 0.141699$ and $M3 = 0.243966$. The earlier near-equality of 0.455 and 0.454 is not reproduced and is withdrawn as evidence of consistency. Although error cancellation can occur in principle, it is not assigned as the explanation of values unsupported by the audited calculations.

The effect of thickness remains method-dependent. At $L/H = 3.5$, changing t from 1 to 2 mm changes $M1$ from 0.129394 to 0.141699 and $M2$ from 0.234305 to 0.236051, while $M3$ changes from 0.244384 to 0.243966. This supports a weaker thickness sensitivity of the sectional estimate within the tested range, without establishing its independence from member length or end loading.

The Sigma-section results therefore cannot be summarized by treating the identification procedure as the only source of variability. The interaction among the procedure, wall thickness, member slenderness, and coupled open-section deformation is significant. For this reason, the most demanding configuration, with $L = 350$ mm and $t = 2$ mm, is examined further through the multi-level mesh convergence study (Fig. 7).

3.3.2. Multi-level mesh-convergence study

A dedicated mesh-convergence study is performed for the Sigma-section with $L = 350$, $L/H = 3.5$, $t = 2$ mm. Four nominal element sizes are considered: $h_e = 5, 3, 2$, and 1 mm.

This configuration was selected to examine the discretization sensitivity of a geometrically complex open profile and whether method disagreement persists under refinement. The four mesh levels are evaluated using the same implementation, geometry, and loading convention. The tabulated results are obtained directly from the solved configurations.

The number of assembled physical degrees of freedom, reaction force, three identified parameters, and their relative deviations from the 1 mm reference mesh are summarized in Table 5. Prescribed degrees of freedom are included in the stated assembled count; the free counts are supplied separately in the numerical files.

Relative to the 1 mm reference, the 2 mm mesh gives deviations of 4.854%, 0.694%, and 0.136% for $M1$, $M2$, and $M3$. At 3 mm, the corresponding deviations are 11.020%, 1.483%, and 0.261%; at 5 mm they are 26.659%, 2.939%, and 0.435%. The markedly different sensitivities require separate convergence statements for the three procedures.

The 2 mm discretization meets the adopted 1% mesh-change criterion for $M2$ and $M3$, but not for $M1$. The finest solution remains a numerical reference rather than an exact continuum result. The reaction decreases from 7320.642 to 6579.342 N under refinement, and $M1$ decreases consistently from 0.141699 to 0.111874. This trend satisfies the

monotonic relation between reaction and $M1$ for fixed geometry and prescribed displacement.

On the finest mesh, $M1 = 0.111874$ and $M3 = 0.245033$ differ by 54.34% relative to $M3$, clearly exceeding the final observed mesh change. By contrast, $M2 = 0.243199$ differs from $M3$ by only about 0.75%, comparable to the 0.694% final change in $M2$. The latter difference cannot confidently be assigned entirely to a physical mechanism. Observed mesh changes are not rigorous bounds on discretization uncertainty.

3.4. C-section: influence of open-section torsion and warping

The C-section is an open, monosymmetric thin-walled profile with an overall depth $H = 100$ mm, two flanges of length 60 mm, two edge lips of length 10 mm, and a uniform wall thickness t . Unlike the doubly symmetric I-section, the C-section has its shear centre located outside the material contour. Transverse loading may therefore activate coupled bending and torsion unless the resultant force acts through the shear centre and the cross-section is free to develop the corresponding warping field.

In the present displacement-controlled benchmark, the same vertical displacement is imposed at all loaded-end nodes. This avoids a concentrated nodal force but imposes an end-section kinematic constraint. The response may contain restrained torsion, axial non-planarity, and flange or lip distortion in addition to transverse shear. Section 3.9.2 compares this arrangement with distributed-force loading, while direct fields are used to support rather than presume the presence of additional deformation.

The geometric proportions also provide a useful first-order interpretation of the sectional result. Using the wall midline dimensions, the total material area is proportional to $A = t(100 + 2 \cdot 60 + 2 \cdot 10) = 240t$, whereas the web area is $A_w = 100t$. The corresponding web-to-total-area ratio is $A_w/A = 100/240 = 0.4167$. This ratio is not an analytical shear correction factor because the flanges and lips also participate in shear transfer and the actual distribution is nonuniform. It nevertheless provides a useful geometric reference for a section in which transverse shear is carried predominantly by the web.

The shell model, cross-sectional geometry, and identified effective shear parameters are presented in Fig. 8.

The four member lengths correspond to $L/H = 0.5, 1.5, 2.5$, and 3.5 . The numerical values are summarized in Table 6.

$M3$ varies between 0.391479 and 0.403876 in the baseline campaign. These values lie below the geometric web-to-total-area ratio 0.4167, as expected when the nonuniform shear field and energy carried

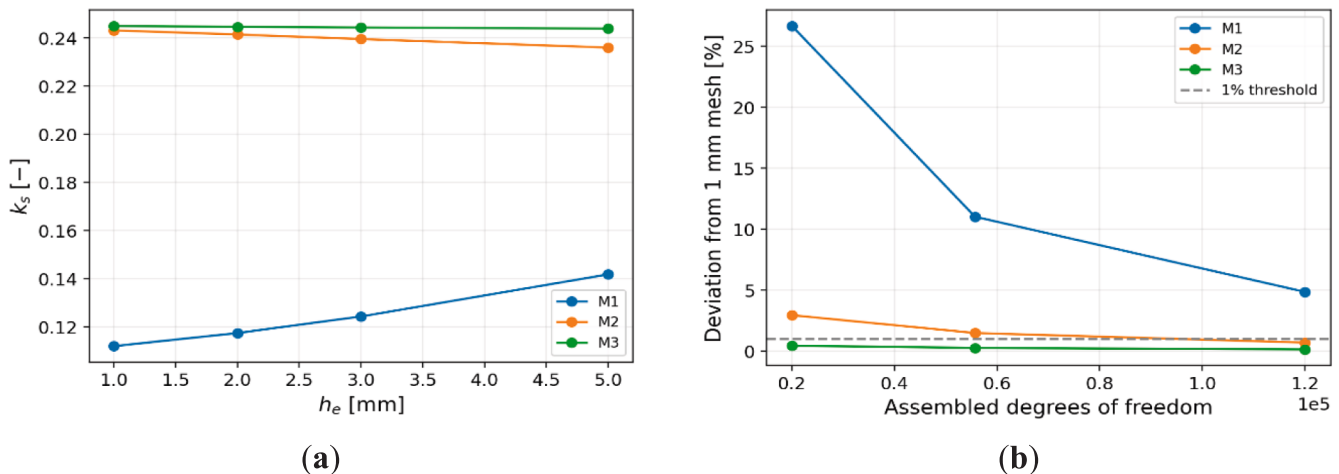


Fig. 7. Executed multi-level mesh study for Sigma with $L = 350$ mm and $t = 2$ mm: (a) identified parameters versus h_e ; (b) deviations from the 1 mm reference versus the assembled physical degrees of freedom, with the 1% practical threshold. The finest point has zero deviation by definition and is omitted from panel (b).

Table 5

Multi-level mesh-convergence results for the Sigma-section with $L = 350$ mm, $L/H = 3.5$, and $t = 2$ mm. The relative deviations $\eta_h^{(m)}$ are calculated with respect to the $h_e = 1$ mm solution.

h_e [mm]	DOFs [-]	V [N]	$k_s^{(1)}$ [-]	$\eta_h^{(1)}$ [%]	$k_s^{(2)}$ [-]	$\eta_h^{(2)}$ [%]	$k_s^{(3)}$ [-]	$\eta_h^{(3)}$ [%]
5	20,022	7320.6	0.141699	26.659	0.236051	2.939	0.243966	0.435
3	55,696	6909.3	0.124203	11.020	0.239593	1.483	0.244393	0.261
2	120,032	6729.2	0.117305	4.854	0.241511	0.694	0.244699	0.136
1	471,042	6579.3	0.111874	0.000	0.243199	0.000	0.245033	0.000

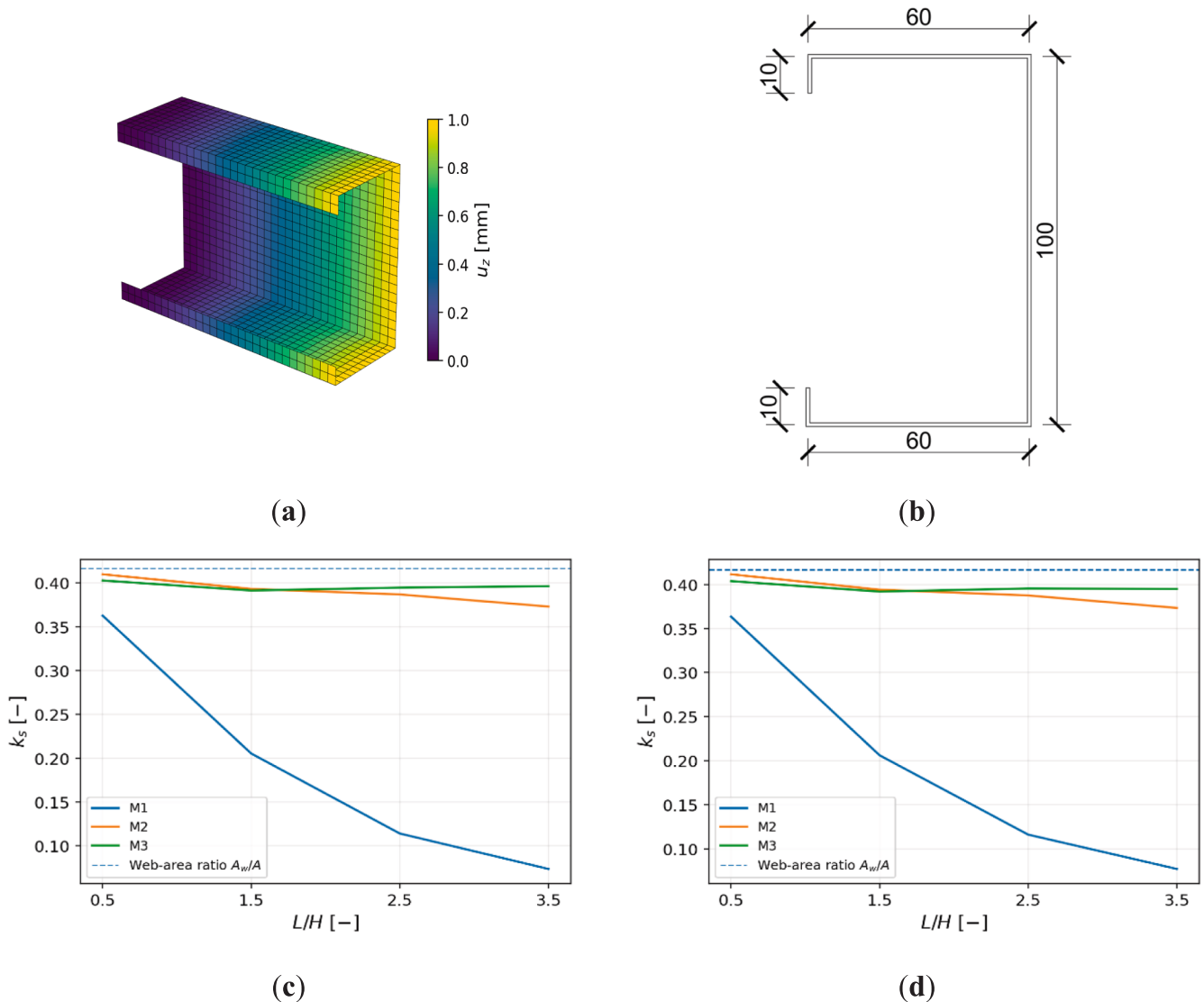


Fig. 8. Geometry, representative shell finite element model, and effective shear parameters for the C-section: (a) shell model with the transverse displacement field; (b) cross-sectional midline geometry and dimensions in millimeters, with $H = 100$ mm, flange length $B = 60$ mm, and lip length $c = 10$ mm; (c) results for $t = 1$ mm and $h_e = 5$ mm; and (d) results for $t = 2$ mm and $h_e = 5$ mm. The horizontal reference line represents the geometric web-to-total-area ratio $A_w/A = 0.4167$ and is not an exact analytical shear correction factor. Panel (a) shows the executed displacement field on the undeformed mesh for $L = 150$ mm, $t = 2$ mm, and $h_e = 5$ mm.

by the flanges and lips are retained. The relatively narrow baseline range is a numerical observation within the prescribed end configuration; it does not establish an intrinsic sectional constant.

For $t = 2$ mm, M3 changes from 0.403876 at $L/H = 0.5$ to 0.392052, 0.395559, and 0.395041 at the larger baseline ratios. The variation is not monotonic. The extended-length results in Section 3.9.1 further reduce the mid-length value to 0.350798 and 0.318645 at $L/H = 7$ and 14, despite a small within-member central range.

Method 2 decreases with member length: from 0.409998 to 0.373105 for $t = 1$ mm, and from 0.411681 to 0.373444 for $t = 2$ mm. It

lies slightly above M3 for the shortest members and below it for the longer ones. Thus the effect of full-volume integration cannot be summarized as a systematic upward shift of the identified factor.

The sign and magnitude of the M2–M3 difference reflect their different spatial domains and recovery rules, in addition to any boundary influence on the solved field. A difference between scalar values alone does not quantify flange–lip distortion or restrained warping. The direct stress and deformation evidence in Section 3.9.2 is therefore considered alongside the mesh comparison in Section 3.9.3.

Method 1 is substantially below the two field-based measures for the

Table 6Reaction force and effective shear parameters identified for the C-section using $h_e = 5$ mm.

L [mm]	L/H [-]	t [mm]	h_e [mm]	V [N]	$k_s^{(1)}$ [-]	$k_s^{(2)}$ [-]	$k_s^{(3)}$ [-]
50	0.5	1	5	131,961	0.362777	0.409998	0.402871
150	1.5	1	5	19,874	0.205355	0.393311	0.391479
250	2.5	1	5	5,827	0.114016	0.386988	0.394733
350	3.5	1	5	2,464	0.073599	0.373105	0.396461
50	0.5	2	5	264,491	0.363612	0.411681	0.403876
150	1.5	2	5	39,859	0.206121	0.394174	0.392052
250	2.5	2	5	11,790	0.116042	0.387655	0.395559
350	3.5	2	5	5,068	0.077130	0.373444	0.395041

longer members. For $t = 1$ mm it decreases from 0.362777 at $L/H = 0.5$ to 0.073599 at 3.5. For $t = 2$ mm the corresponding values are 0.363612 and 0.077130. This strong length dependence is not reproduced by the baseline mid-length M3 values.

A lower Method 1 value corresponds to a larger displacement residual after subtraction of the Euler–Bernoulli bending contribution. For the C-section, this residual cannot be attributed uniquely to transverse shear. Restrained torsion, nonuniform warping, and local cross-sectional distortion may increase the global compliance and are consequently incorporated into the effective parameter obtained using Method 1.

The M1–M3 separation increases strongly over the baseline length range. At $t = 2$ mm it changes from approximately -9.97% relative to M3 at $L/H = 0.5$ to -80.48% at 3.5. The conditioning analysis shows that a large method difference can coexist with modest amplification of bending-reference uncertainty; mechanical incompleteness of the scalar reduction must therefore be distinguished from numerical subtraction sensitivity.

The C-section demonstrates why a locally stable sectional estimate and a transferable section property are different claims. M3 is comparatively stable within the baseline campaign, but changes under length extension and alternative end loading. M1 remains a configuration-dependent measure of the complete constrained response, while M2 combines transverse shear energy throughout the member with sensitivity to raw Gauss-point fields.

3.5. Z-Section: breakdown of the single-axis global bending decomposition

The Z-section is an open thin-walled profile with an overall depth $H = 100$ mm, two oppositely directed flanges of length 40 mm, two edge lips of length 10 mm, and a uniform wall thickness t . The profile is point-symmetric but does not possess reflection symmetry with respect to the global y - or z -axis.

This geometry is particularly important for assessing the global compliance procedure. In the adopted global coordinate system, the product of inertia I_{yz} is nonzero; consequently, the y - and z -axes are not principal bending axes. A transverse force acting in the global z -direction produces coupled unsymmetric bending rather than a response governed solely by EI_y .

For a centroidal coordinate system with $I_{yz} \neq 0$, the flexural response associated with a resultant bending moment about the global y -axis is governed by the complete sectional inertia matrix. When the resultant moment about the global z -axis vanishes, the effective flexural rigidity associated with transverse displacement in the global z -direction can be expressed as

$$EI_{b,\text{eff}} = E \left(I_y - \frac{I_{yz}^2}{I_z} \right),$$

rather than simply EI_y . Consequently, the Euler–Bernoulli term $VL^3/(3EI_y)$ used in Method 1 does not reproduce the complete bending compliance of this section.

The incomplete single-axis bending representation does not enter Methods 2 and 3 in the same way because these procedures are based on the recovered beam-oriented shear stress and strain fields rather than on

subtraction of an analytical bending displacement. The Z-section therefore provides a clear example of a geometry for which the scalar global bending-plus-shear decomposition underlying Method 1 is mechanically insufficient.

A first-order geometric reference can also be obtained from the wall areas. For the adopted midline dimensions, $A = t(100 + 2 \cdot 40 + 2 \cdot 10) = 200t$, whereas the web area is $A_w = 100t$. Thus, $A_w/A = 0.5$. This ratio is not an analytical shear correction factor, but it provides a useful reference for interpreting the sectional result when shear transfer is dominated by the vertical web.

The geometry, representative shell model, and identified effective shear parameters are presented in Fig. 9.

The four member lengths correspond to $L/H = 0.5, 1.5, 2.5$, and 3.5. The corresponding numerical values are reported in Table 7.

Method 3 changes from 0.488070–0.488804 at $L/H = 0.5$ to 0.471414–0.471525 at 3.5, with a weak thickness dependence. Its proximity to, but difference from, the geometric ratio 0.5 is consistent with predominantly web-controlled shear transfer and the contribution of the flanges and lips to the complete transverse energy.

Method 2 decreases from 0.495116 to 0.452649 for $t = 1$ mm and from 0.496732 to 0.453247 for $t = 2$ mm. At the longest member, it is approximately 3.9% below M3. Their difference is therefore not eliminated by increasing the baseline length, and should be considered together with field recovery and the finite-member configuration.

Method 1 behaves fundamentally differently. It decreases from 0.437670 to 0.438511 at $L/H = 0.5$ to 0.234843–0.235168 at 1.5, 0.122379–0.122541 at 2.5, and 0.071172–0.071265 at 3.5. The weak thickness dependence supports a geometrical and kinematic origin for this trend rather than a new material shear mechanism.

In Method 1, the shear displacement is calculated as the difference between the shell-model displacement and the single-axis reference bending displacement $VL^3/(3EI_y)$. For the Z-section, the nonzero product of inertia I_{yz} produces unsymmetric bending, so the actual bending compliance is not represented by EI_y alone. The resulting bending-compliance discrepancy scales with L^3 and is assigned to the displacement residual interpreted as shear. Because the identified shear stiffness is proportional to $L/w_s^{(1)}$, this contamination causes $k_s^{(1)}$ to decrease rapidly as member length increases.

Additional effects associated with end kinematics, restrained warping, and torsional response may also contribute to the displacement. The nonzero product of inertia establishes that the single-axis bending reference is incomplete, but the method differences do not uniquely partition the residual among these effects.

The very low global values must not be interpreted as physical sectional shear correction factors. They demonstrate the breakdown of Method 1 when applied to a section whose bending response requires the complete sectional inertia matrix. For such a profile, either the coupled bending contribution must be removed using the full unsymmetric-bending formulation or the effective shear stiffness must be identified using a field-based procedure.

The Z-section shows that correspondence between a global cantilever response and a scalar Timoshenko shear parameter cannot be assumed for arbitrary thin-walled geometry. M3 remains near 0.47 for the longer

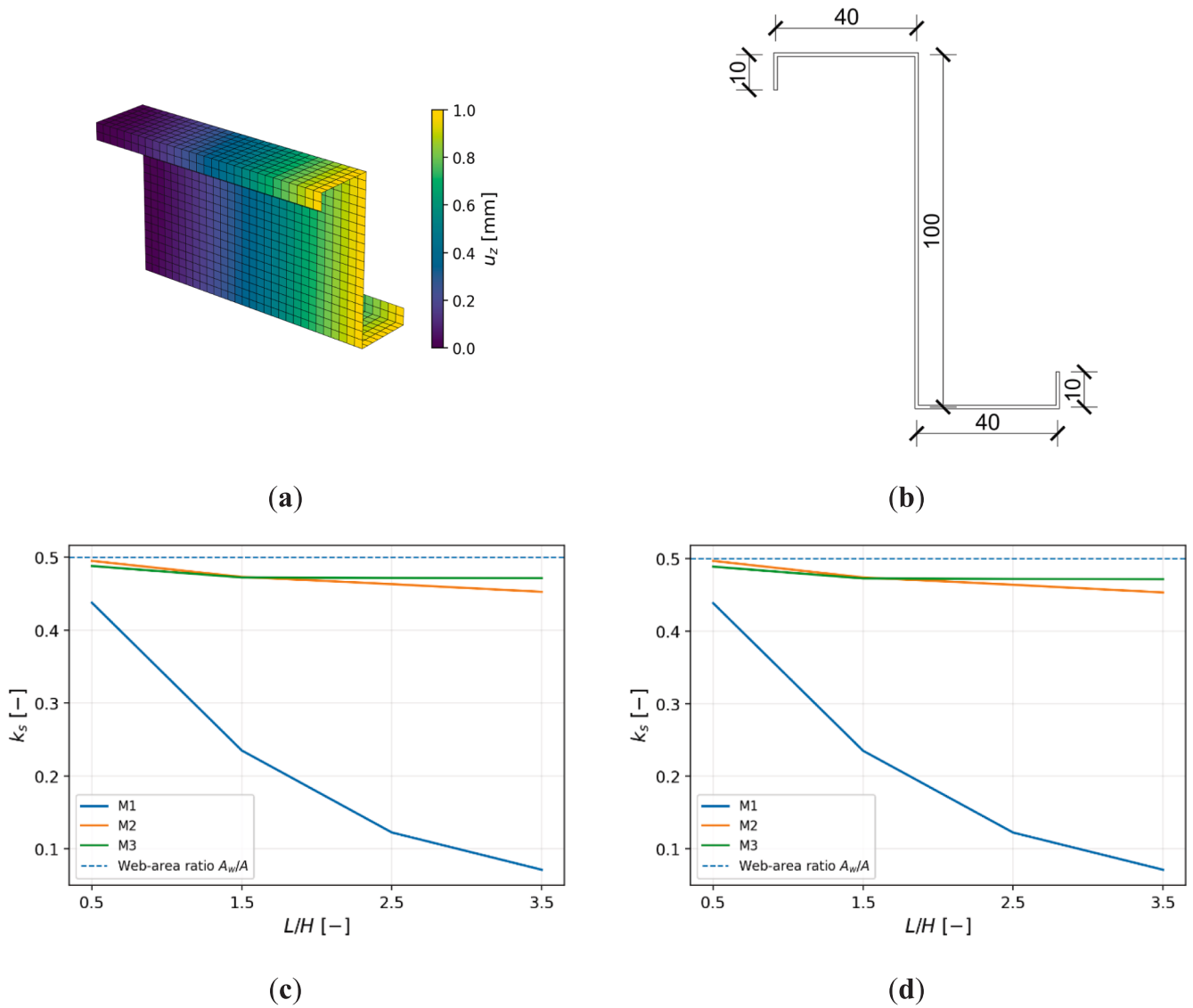


Fig. 9. Geometry, representative shell finite element model, and effective shear parameters for the Z-section: (a) shell model with the transverse displacement field; (b) cross-sectional midline geometry and dimensions in millimeters, with $H = 100$ mm, flange length $B = 40$ mm, and lip length $c = 10$ mm; (c) results for $t = 1$ mm and $h_e = 5$ mm; and (d) results for $t = 2$ mm and $h_e = 5$ mm. The horizontal reference line represents the geometric web-to-total-area ratio $A_w/A = 0.5$ and is not an exact analytical shear correction factor. Panel (a) shows the executed displacement field on the undeformed mesh for $L = 150$ mm, $t = 2$ mm, and $h_e = 5$ mm.

Table 7

Reaction force and effective shear parameters identified for the Z-section using $h_e = 5$ mm.

L [mm]	L/H [-]	t [mm]	h_e [mm]	V [N]	$k_s^{(1)}$ [-]	$k_s^{(2)}$ [-]	$k_s^{(3)}$ [-]
50	0.5	1	5	130,133	0.437670	0.495116	0.488070
150	1.5	1	5	17,833	0.234843	0.473120	0.472477
250	2.5	1	5	4,926	0.122379	0.463239	0.471728
350	3.5	1	5	1,944	0.071172	0.452649	0.471414
50	0.5	2	5	260,726	0.438511	0.496732	0.488804
150	1.5	2	5	35,702	0.235168	0.473943	0.472657
250	2.5	2	5	9,860	0.122541	0.463909	0.471842
350	3.5	2	5	3,890	0.071265	0.453247	0.471525

baseline members, while M1 identifies a strongly length-dependent residual containing an incompletely represented bending contribution and potentially other coupled deformation modes.

3.6. Omega-section: symmetric open-section response

The Omega-section is a symmetric open thin-walled profile consist-

ing of a top horizontal wall, two parallel vertical webs, two outward bottom flanges, and two edge lips. The section has an overall depth $H = 100$ mm, a top-wall width of 40 mm, bottom-flange lengths of 20 mm, edge-lip lengths of 10 mm, and a uniform wall thickness t .

In contrast to the C- and Z-sections, the Omega-section is symmetric with respect to the vertical loading plane. The product of inertia I_{yz} vanishes when the centroidal symmetry axes are used, and transverse

loading along the symmetry axis does not activate the same unsymmetric-bending mechanism observed for the Z-section. The two vertical webs also provide parallel and geometrically balanced shear-transfer paths.

The profile nevertheless remains open. Torsional warping, local bending of the top wall and bottom flanges, and distortion of the web-flange junctions may therefore contribute to the shell response, particularly near the clamped and displacement-controlled ends.

A first-order geometric reference can be obtained by comparing the total area of the two vertical webs with the complete material area. Using the wall midline dimensions, $A = t(40 + 2 \cdot 100 + 2 \cdot 20 + 2 \cdot 10) = 300t$, whereas the combined area of the two vertical webs is $A_w = 2 \cdot 100t = 200t$. Therefore, $A_w/A = 200/300 = 0.6667$. This ratio is not an analytical shear correction factor. It provides only a geometric reference for a section in which transverse shear is expected to be transferred primarily through the two vertical webs.

The geometry, representative shell finite element model, and

identified effective shear parameters are presented in Fig. 10.

The four analyzed lengths correspond to $L/H = 0.5, 1.5, 2.5$, and 3.5 . The numerical values are summarized in Table 8.

Method 3 is relatively stable compared with M1 but not constant. For $t = 1$ mm its values are 0.635033, 0.617428, 0.623792, and 0.619408 as L/H increases. For $t = 2$ mm they are 0.635493, 0.618085, 0.622220, and 0.614475. The non-monotonic variation cautions against assuming that geometric symmetry alone establishes a developed sectional state.

The sectional results lie below the two-web area ratio 0.6667. This is consistent with nonuniform wall shear, redistribution at web-top-wall and web-flange junctions, and the contribution of horizontal-wall shear to the complete energy. The area ratio remains a qualitative geometric indicator, not a reference solution.

Method 2 decreases from 0.644018 to 0.590053 for $t = 1$ mm and from 0.645580 to 0.594007 for $t = 2$ mm. It is close to M3 at $L/H = 1.5$, but lower than M3 at the two larger lengths. Full-volume and recovered sectional energies therefore remain distinct for this symmetric open

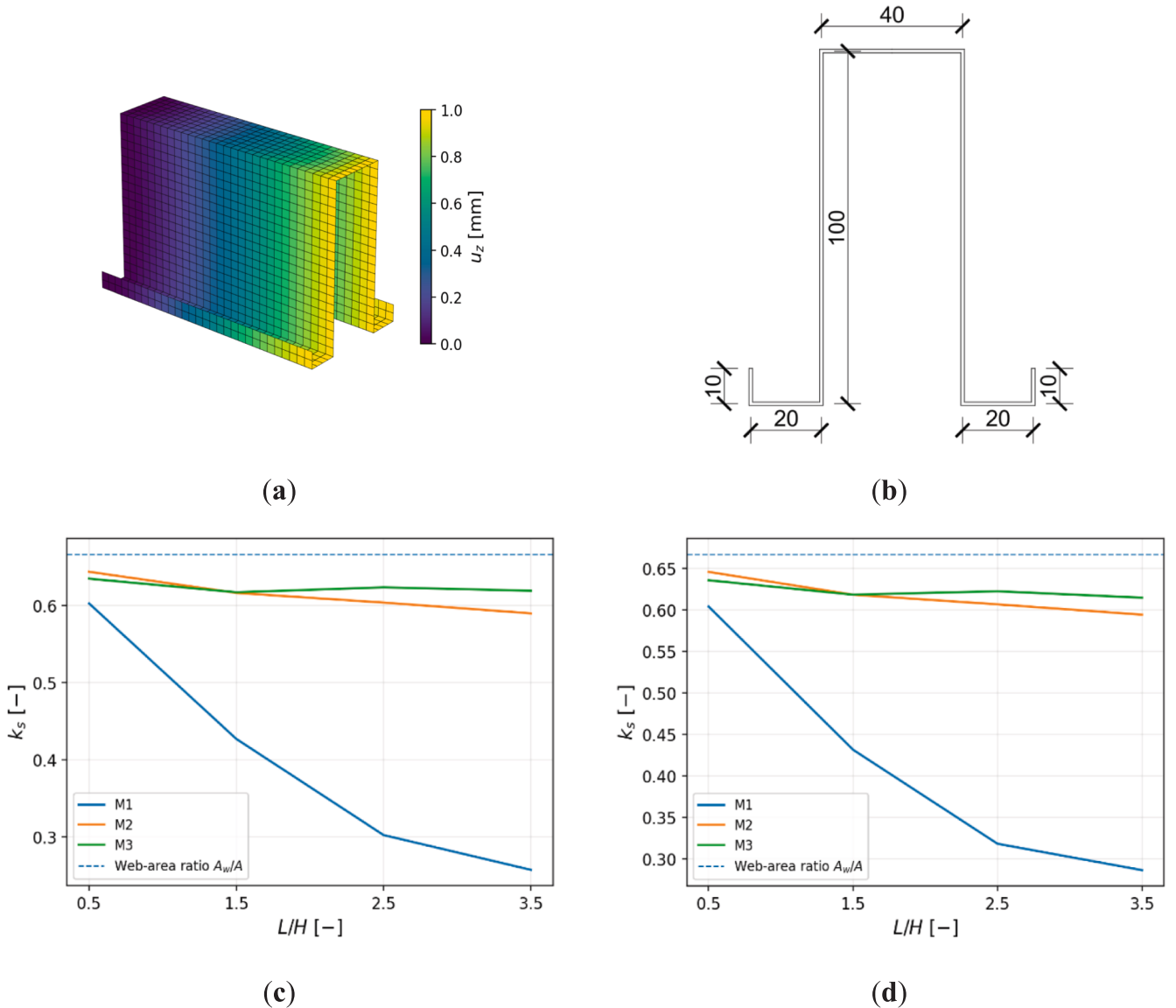


Fig. 10. Geometry, representative shell finite element model, and effective shear parameters for the Omega-section: (a) shell model with the transverse displacement field; (b) cross-sectional midline geometry and dimensions in millimeters, with $H = 100$ mm, top-wall width $B_{top} = 40$ mm, bottom-flange lengths $B_f = 20$ mm, and lip lengths $c = 10$ mm; (c) results for $t = 1$ mm and $h_e = 5$ mm; and (d) results for $t = 2$ mm and $h_e = 5$ mm. The horizontal reference line represents the geometric two-web-to-total-area ratio $A_w/A = 0.6667$ and is not an exact analytical shear correction factor. Panel (a) shows the executed displacement field on the undeformed mesh for $L = 150$ mm, $t = 2$ mm, and $h_e = 5$ mm.

Table 8Reaction force and effective shear parameters identified for the Omega-section using $h_e = 5$ mm.

L [mm]	L/H [-]	t [mm]	h_e [mm]	V [N]	$k_s^{(1)}$ [-]	$k_s^{(2)}$ [-]	$k_s^{(3)}$ [-]
50	0.5	1	5	255,548	0.602866	0.644018	0.635033
150	1.5	1	5	36,048	0.426888	0.616602	0.617428
250	2.5	1	5	10,476	0.302051	0.604022	0.623792
350	3.5	1	5	4,456	0.256942	0.590053	0.619408
50	0.5	2	5	511,854	0.603888	0.645580	0.635493
150	1.5	2	5	72,479	0.431248	0.618014	0.618085
250	2.5	2	5	21,334	0.317923	0.606453	0.622220
350	3.5	2	5	9,146	0.286133	0.594007	0.614475

profile.

At $L/H = 3.5$, the M2–M3 difference is -0.029355 for $t = 1$ mm and -0.020468 for $t = 2$ mm, corresponding to about -4.74% and -3.33% relative to M3. These are observed method differences, not estimates of a separately identified distortion-energy contribution.

The Omega-section does not require the unsymmetric-bending correction associated with nonzero I_{yz} . Nevertheless, M1 decreases from 0.602866 to 0.256942 for $t = 1$ mm and from 0.603888 to 0.286133 for $t = 2$ mm. The reconstructed results therefore show substantial global-sectional disagreement even in this symmetric open section.

For the longest, thicker-walled member, M1 = 0.286133 and M3 = 0.614475 differ by approximately -53.43% relative to M3. This separation is not a physical decrease in the steel shear modulus and cannot be removed merely by observing that $I_{yz} = 0$.

The prescribed end motion and local shell deformation affect the complete compliance. The global residual can consequently contain contributions omitted by the assumed bending-plus-shear reference. Local wall bending and sectional distortion are plausible contributions, but a quantitative allocation would require additional generalized coordinates or an independent modal energy decomposition. The present scalar comparison does not provide that allocation.

The Omega-section remains an intermediate geometrical case between monosymmetric open profiles and fully closed contours. Its symmetry suppresses the particular unsymmetric-bending mechanism of the Z-section, but does not make the three extraction procedures equivalent. The result reinforces the need to test compatibility between the detailed shell response and the retained beam kinematics.

The M3 values around 0.61–0.64 characterize the investigated finite-member configurations. They should not be promoted to universal design constants without additional length, boundary, and mesh checks for this profile. Method 1 requires particular caution where it differs substantially from both field-based estimates.

3.7. Rectangular hollow section: closed-contour shear transfer

The rectangular hollow section is a doubly symmetric, fully closed thin-walled profile with an overall height $H = 100$ mm, an overall width $B = 60$ mm, and a uniform wall thickness t . Its centroid, shear centre, and principal-axis intersection coincide. Consequently, transverse loading in the global z -direction does not activate the unsymmetric-bending mechanism identified for the Z-section.

The closed contour provides a continuous shear-flow path and substantially greater torsional restraint than the previously considered open sections. Nonuniform warping is also more strongly constrained by the geometry of the closed cell. These properties are expected to improve the stability of shear-stiffness identification, particularly for methods based directly on sectional or volume-integrated shear fields.

A first-order geometric reference can be constructed from the areas of the two walls aligned primarily with the applied shear direction. Using the wall midline dimensions, the total material area is $A = t(2H + 2B) = t(2 \cdot 100 + 2 \cdot 60) = 320t$, whereas the combined area of the two vertical walls is $A_w = 2Ht = 200t$. Therefore, $A_w/A = 200/320 = 0.625$. This ratio is not an exact analytical shear correction factor. It

provides a geometric reference for a closed section in which the two vertical walls carry the dominant part of the transverse shear, while the horizontal walls complete the closed shear-flow path and contribute to local redistribution.

The geometry, representative shell finite element model, and identified effective shear parameters are presented in Fig. 11.

The four analyzed member lengths correspond to $L/H = 0.5, 1.5, 2.5$, and 3.5 . The corresponding numerical values are summarised in Table 9.

Method 3 ranges from 0.562333 to 0.577073 over the rectangular hollow-section campaign, with a small thickness dependence. The shortest members give the largest values; the longer configurations lie near 0.565. This relative stability is consistent with the continuous closed shear-flow path, while still reflecting finite-member and recovery effects.

The sectional values are moderately lower than the geometric vertical-wall area ratio $A_w/A = 0.625$. This difference is mechanically consistent with the nonuniform shear distribution in the two vertical walls, the transfer of shear flow through the corner regions, and the participation of the horizontal walls in maintaining the closed-cell response. The geometric ratio should therefore be treated as a first-order area reference rather than as an exact sectional solution.

Method 2 decreases from 0.589674 to 0.561445 for $t = 1$ mm and from 0.591332 to 0.562033 for $t = 2$ mm as L/H increases from 0.5 to 3.5. The variation is modest, but the result is not strictly length-independent.

The relatively small separation between M2 and M3 in the longer configurations supports consistency of the volume and sectional measures for this closed profile. It does not establish the absence of every local wall or corner deformation mechanism, because those mechanisms have not been isolated energetically.

Method 2 is slightly below M3 at $L/H = 3.5$: the absolute differences are -0.003262 for $t = 1$ mm and -0.002748 for $t = 2$ mm. The integrated measure is therefore not systematically above the sectional one.

Method 1 increases moderately with member slenderness. For $t = 1$ mm it changes from 0.574220 to 0.597199; for $t = 2$ mm, from 0.575171 to 0.598259. These values remain below the geometric wall-area ratio, although the longest-member M1 exceeds the two field-based estimates.

The rectangular hollow section is doubly symmetric, has $I_{yz} = 0$, and does not exhibit the incomplete unsymmetric-bending subtraction responsible for the collapse of Method 1 in the Z-section. The observed increase must therefore have a different origin. As the member becomes longer, the total displacement is increasingly dominated by bending, while the shear displacement used in Method 1 is obtained as the small residual – see Eq. (21).

A mismatch between shell-model bending compliance and the Euler–Bernoulli reference can cause a larger relative change in the residual shear displacement. The sensitivity is quantified directly in Section 3.9.5. The rectangular-section results show only a weak thickness dependence over the investigated range.

The difference between M1 and the field-based values is interpreted as a combination of the assumed flexural reference, finite-member response, and subtraction sensitivity. It is not used as evidence of a

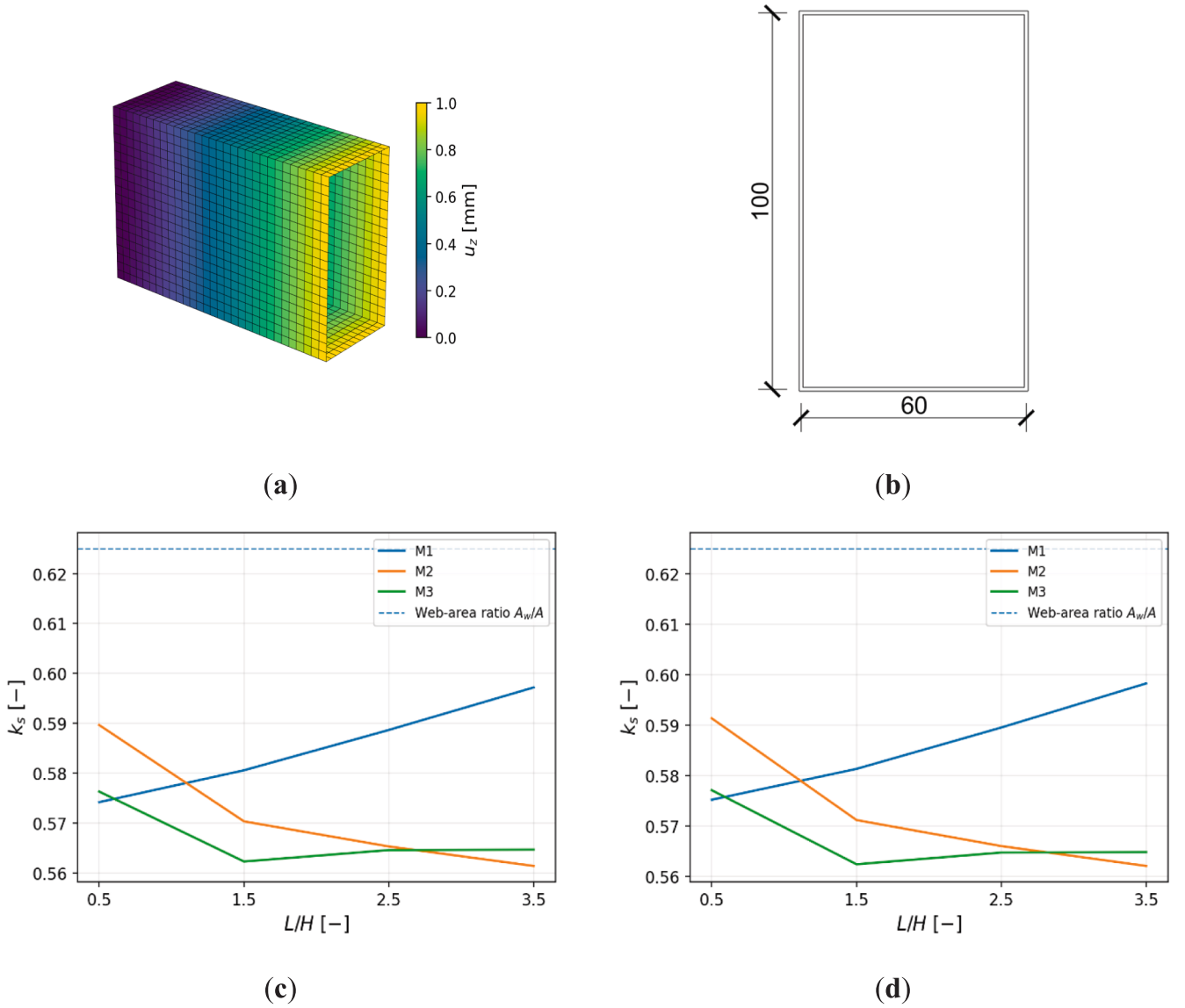


Fig. 11. Geometry, representative shell finite element model, and effective shear parameters for the rectangular hollow section: (a) shell model with the transverse displacement field; (b) cross-sectional midline geometry and dimensions in millimeters, with $H = 100$ mm, $B = 60$ mm, and wall thicknesses $t = 1$ and 2 mm; (c) results for $t = 1$ mm and $h_e = 5$ mm; and (d) results for $t = 2$ mm and $h_e = 5$ mm. The horizontal reference line represents the geometric vertical-wall-to-total-area ratio $A_w/A = 0.625$ and is not an exact analytical shear correction factor. Panel (a) shows the executed displacement field on the undeformed mesh for $L = 150$ mm, $t = 2$ mm, and $h_e = 5$ mm.

Table 9

Reaction force and effective shear parameters identified for the rectangular hollow section using $h_e = 5$ mm.

L [mm]	L/H [-]	t [mm]	h_e [mm]	V [N]	$k_s^{(1)}$ [-]	$k_s^{(2)}$ [-]	$k_s^{(3)}$ [-]
50	0.5	1	5	263,565	0.574220	0.589674	0.576350
150	1.5	1	5	46,565	0.580607	0.570379	0.562333
250	2.5	1	5	14,373	0.588683	0.565348	0.564625
350	3.5	1	5	5,934	0.597199	0.561445	0.564707
50	0.5	2	5	527,905	0.575171	0.591332	0.577073
150	1.5	2	5	93,182	0.581298	0.571142	0.562368
250	2.5	2	5	28,755	0.589507	0.565974	0.564696
350	3.5	2	5	11,872	0.598259	0.562033	0.564781

length-dependent intrinsic sectional shear stiffness.

The closed rectangular contour gives comparatively consistent field-based estimates, approximately 0.56–0.59 over the baseline campaign. Closure does not make the definitions identical: the global result remains tied to the compliance residual, while M2 and M3 use different spatial

domains and recovery conventions.

The rectangular hollow section therefore supports two separate conclusions. First, closure of the contour produces a repeatable and mechanically interpretable sectional shear response. Second, geometric closure alone does not guarantee reliable global compliance

identification for long, bending-dominated members.

3.8. Circular hollow section: smooth closed contour and thin-wall reference

The circular hollow section is a smooth, doubly symmetric, fully closed thin-walled profile with a nominal radius $R = 50$ mm, corresponding to an overall diameter and characteristic depth $H = 2R = 100$ mm. The section has a uniform wall thickness t , and its centroid, shear centre, and principal-axis intersection coincide. The absence of corners, fold lines, free edges, and geometric discontinuities removes several sources of local field concentration present in the previously analyzed profiles.

The circular section nevertheless remains a shell structure. Local ovalisation, circumferential wall bending, and end-region disturbances may contribute to the numerical response, particularly under the imposed uniform displacement of the loaded end section. Moreover, the orientation of the local shell coordinate system varies continuously

around the circumference. Correct transformation of both the shell stress and strain tensors is therefore particularly important for this geometry.

The circular hollow section also provides a classical thin-wall analytical reference. Let θ denote the circumferential coordinate and let R be the wall midline radius. Under a transverse shear force V , the elementary thin-wall shear-flow distribution can be written as

$$q(\theta) = \frac{V}{\pi R} \cos \theta. \quad (45)$$

The corresponding shear energy per unit member length is

$$U_{\text{ring}} = \oint \frac{q^2}{2Gt} ds = \frac{V^2}{2G\pi R t}. \quad (46)$$

Since the total material area of the thin circular wall is $A = 2\pi R t$, the energy-equivalent shear correction factor becomes

$$k_{s,\text{ring}} = \frac{V^2}{2GAU_{\text{ring}}} = \frac{1}{2}. \quad (47)$$

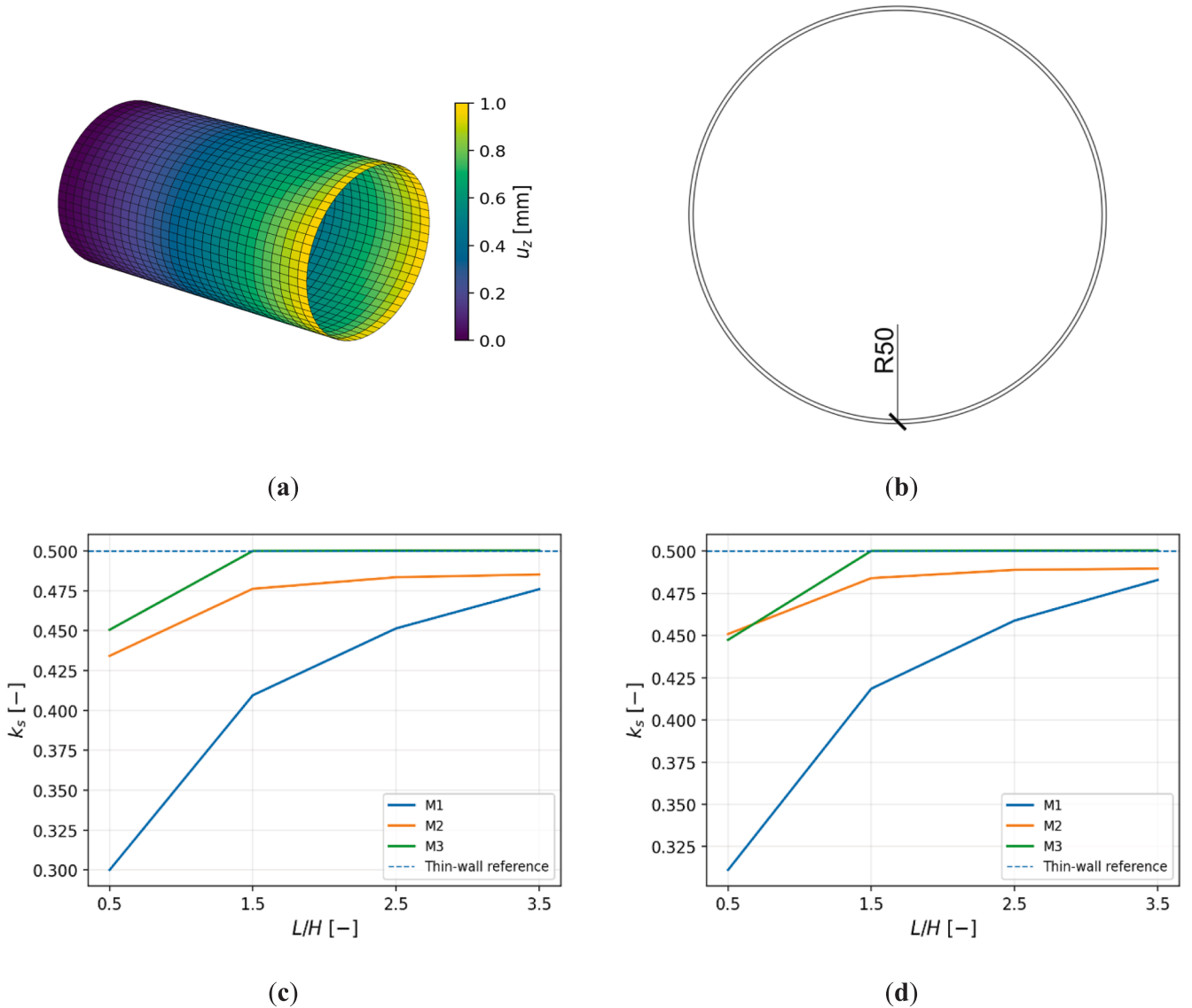


Fig. 12. Geometry, representative shell finite element model, and effective shear parameters for the circular hollow section: (a) shell model with the transverse displacement field; (b) cross-sectional geometry and dimensions in millimeters, with a nominal midline radius $R = 50$ mm, characteristic depth $H = 100$ mm, and wall thicknesses $t = 1$ and 2 mm; (c) results for $t = 1$ mm and $h_e = 5$ mm; and (d) results for $t = 2$ mm and $h_e = 5$ mm. The horizontal reference line represents the classical thin-wall shear-flow result $k_{s,\text{ring}} = 0.5$. Panel (a) shows the executed displacement field on the undeformed mesh for $L = 150$ mm, $t = 2$ mm, and $h_e = 5$ mm.

The value $k_{s,ring} = 0.5$ is an asymptotic thin-wall reference based on an elementary, fully developed shear-flow state. It does not account for finite-length boundary effects, local shell bending, ovalisation, or the specific end-section constraints used in the numerical benchmark.

The geometry, representative shell model, and identified effective shear parameters are presented in Fig. 12.

The four member lengths correspond to $L/H = 0.5, 1.5, 2.5$, and 3.5 . The corresponding numerical values are summarised in Table 10.

Method 3 approaches the elementary thin-wall value rapidly over the baseline range. For $t = 1$ mm, its values are 0.450539, 0.499949, 0.500108, and 0.500252 as L/H increases from 0.5 to 3.5. For $t = 2$ mm, they are 0.447413, 0.500098, 0.500218, and 0.500366. The short-member departure confirms that even this symmetric closed contour is sensitive to the finite benchmark configuration.

For $L/H \geq 1.5$, the recovered complete-energy M3 is close to $k_{s,ring} = 0.5$. At $L/H = 3.5$, $t = 2$ mm, and $h_e = 2$ mm, it is 0.500118, corresponding to a 0.0236% difference from the elementary thin-wall reference. Refinement changes both the finite element spacing and the polygonal approximation of the circular midline.

The depressed sectional values for the shortest member indicate a response not represented by the fully developed thin-wall shear-flow idealization. Local wall deformation and end constraints may contribute, but their individual energies are not separated here. The coordinate transformation is particularly important because the local wall tangent changes around the contour.

Method 2 increases from 0.434163 to 0.485228 for $t = 1$ mm and from 0.450864 to 0.489641 for $t = 2$ mm. It remains below the mid-length M3 values for the longer members. Its approach to the elementary limit is slower than that of the recovered sectional measure because the integration covers the full member and uses raw Gauss-point fields.

The difference is assessed in conjunction with the mesh comparison: at $L = 350$ mm and $t = 2$ mm, refinement from 5 to 2 mm changes M2 by about 1.13% and M3 by about 0.050%. A residual M2–M3 difference of comparable magnitude cannot be attributed entirely to a physical end-region effect.

In the revised implementation, both local tensors are transformed before evaluating the complete work-conjugate transverse product $(\tau_{xy}^{beam}, \tau_{xy}^{beam} + \tau_{xz}^{beam}, \tau_{xz}^{beam})/2$. Retaining only the xz product gives a different energy definition: at $L = 350$ mm, $t = 2$ mm, and $h_e = 5$ mm, the projected M3 is 0.670426 instead of the complete-energy value 0.500366. For the elementary circular shear flow, projection produces integrals of $\cos^4\theta$ rather than the complete $\cos^2\theta$ energy, giving $2/3$ instead of $1/2$. The single-component calculation is retained as a diagnostic in the numerical files; the reported beam shear factor uses the complete transverse energy.

Method 1 increases from 0.299936 to 0.475980 for $t = 1$ mm and from 0.311120 to 0.482903 for $t = 2$ mm. It remains below unity and approaches the sectional value with increasing baseline length. No greater-than-unity value is obtained in this circular benchmark.

Within Eq. (1), a value greater than unity would imply an effective stiffness above the reference GA . Such a result would require careful examination of the identification definition and response compatibility. The current circular benchmark does not exhibit that behavior; the

conditioning limitation of M1 is demonstrated quantitatively using the plate and other representative cases in Section 3.9.5.

The circular section is doubly symmetric and has $I_{yz} = 0$, so unsymmetric bending is not the source of the method differences. The global subtraction can still amplify a mismatch in the flexural reference. This sensitivity is separate from the energy-component error corrected above and from finite-length influences on the shell solution.

The circular benchmark demonstrates the value of an analytical reference whose energy definition matches the numerical one. Complete transverse energy and longitudinal recovery yield a sectional value close to one half, while the raw volume integral and compliance residual approach it at different rates. Smoothness and closure do not by themselves establish method equivalence.

The circular hollow section therefore reinforces the central distinction of the study. A transferable sectional shear parameter must be identified from a mechanically appropriate, sufficiently developed field, whereas a parameter extracted from total cantilever compliance may represent a configuration-dependent residual rather than an intrinsic property of the cross-section.

3.9. Additional verification and boundary-sensitivity results

3.9.1. Longitudinal development of the sectional field

Fig. 13 compares the requested scans for Sigma, C, and circular profiles. For Sigma, the central range is 8.48% at $L/H = 3.5$, 0.844% at 7, and 0.0167% at 14. The original baseline mid-length position is therefore not demonstrated to lie on a plateau. The extended models provide direct evidence of central-field development.

For C, the central range is below 1% at all three lengths, but M3 at mid-length changes from 0.395041 to 0.350798 and 0.318645. A flat field within one finite member is therefore insufficient to establish independence from overall length and restraint. The circular values remain approximately 0.5004, with negligible central variation in the extended cases.

3.9.2. End loading and direct stress and deformation evidence

Table 11 compares uniform displacement (UD) with distributed force (DF). At a common generalized displacement of 1 mm, M3 decreases from 0.244393 to 0.215183 for Sigma (−11.95%) and from 0.394909 to 0.314932 for C (−20.25%). These differences exceed the small observed mesh variation of M3, but characterize the changed loading-and-restraint configuration as a whole.

Fig. 14 supplies direct stress and deformation evidence. Stress is normalized to a common resultant of 1 kN. Distortions are obtained after removing fitted rigid translation and rotation, and refer to a common generalized displacement of 1 mm. For Sigma, the near-mid-length distortion measure changes from 0.00917 to 0.07258 mm; for C, from 0.02619 to 0.03646 mm. These are not fixed-force deformation comparisons.

The supplied diagnostics also show nonzero twist and axial non-planarity. They support a response richer than planar bending plus shear, without quantifying the allocation of M1 residual energy among torsion, warping, and distortion. The distinction between observed fields and a proposed causal interpretation is retained.

Table 10

Reaction force and effective shear parameters identified for the circular hollow section using $h_e = 5$ mm.

L [mm]	L/H [-]	t [mm]	h_e [mm]	V [N]	$k_s^{(1)}$ [-]	$k_s^{(2)}$ [-]	$k_s^{(3)}$ [-]
50	0.5	1	5	141,271	0.299936	0.434163	0.450539
150	1.5	1	5	35,573	0.409491	0.476252	0.499949
250	2.5	1	5	11,748	0.451389	0.483485	0.500108
350	3.5	1	5	4,935	0.475980	0.485228	0.500252
50	0.5	2	5	292,297	0.311120	0.450864	0.447413
150	1.5	2	5	71,942	0.418507	0.483987	0.500098
250	2.5	2	5	23,595	0.458829	0.488881	0.500218
350	3.5	2	5	9,890	0.482903	0.489641	0.500366

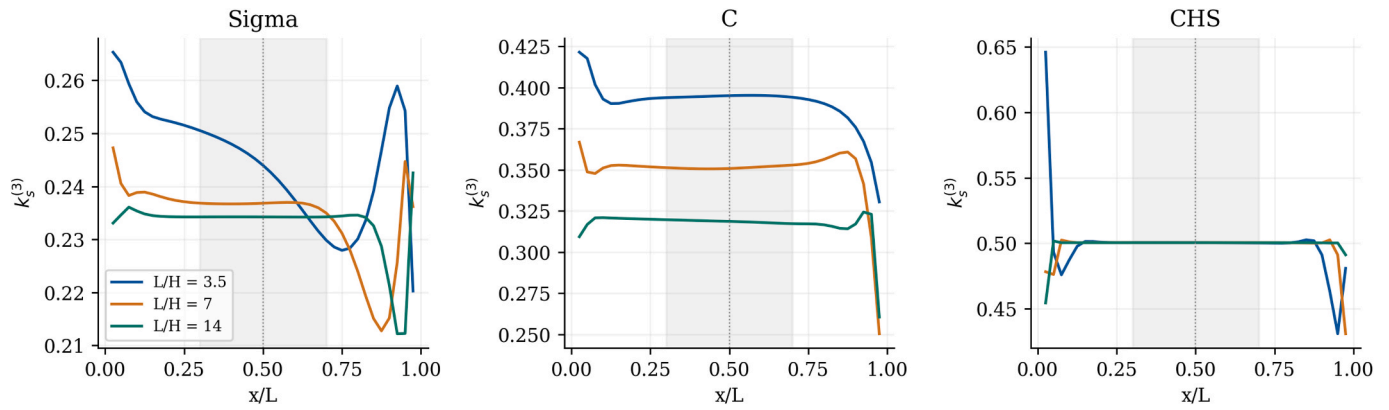


Fig. 13. Longitudinal M3 scans for Sigma, C, and circular sections at $t = 2$ mm and $h_e = 5$ mm. The shaded central 40% is the region used to measure spatial variation; it does not assert that a plateau exists.

Table 11

End-condition comparison at $L = 350$ mm, $t = 2$ mm, and $h_e = 3$ mm. Both solutions use a 1 mm work-conjugate generalized displacement.

Section	End	V [N]	M1	M2	M3
Sigma	UD	6909.294	0.124203	0.239593	0.244393
Sigma	DF	1607.810	0.015063	0.203108	0.215183
C	UD	4894.358	0.072768	0.383469	0.394909
C	DF	1441.499	0.014716	0.280615	0.314932

3.9.3. Method disagreement relative to mesh sensitivity

Table 12 gives method differences relative to M3 for additional representative sections, evaluated on the 2 mm mesh, together with changes from 5 to 2 mm. These are observed refinement changes rather than continuum uncertainty bounds. The C-section M1–M3 separation clearly exceeds the field-based mesh sensitivity, whereas several M2–M3

differences are comparable with the M2 mesh change.

For Sigma, the four-level evidence in Table 5 supports the same distinction: the finest M1–M3 difference is about -54.34% , whereas the M2–M3 difference is approximately -0.75% . Only the former clearly dominates the final observed refinement change. Attribution of the smaller difference wholly to physical coupling would overstate the available evidence.

3.9.4. Influence of the local shell shear coefficient

Table 13 reports the $\pm 20\%$ variation of the local coefficient. Relative to 5/6, M3 changes by $+0.0520\%$ and -0.0434% for Sigma, and by -0.0118% and $+0.0059\%$ for C. M2 changes by less than 0.26%, while M1 changes by up to 2.23%. The tested sectional results are weakly sensitive to this modelling choice, without establishing a universal conclusion for curved or anisotropic walls.

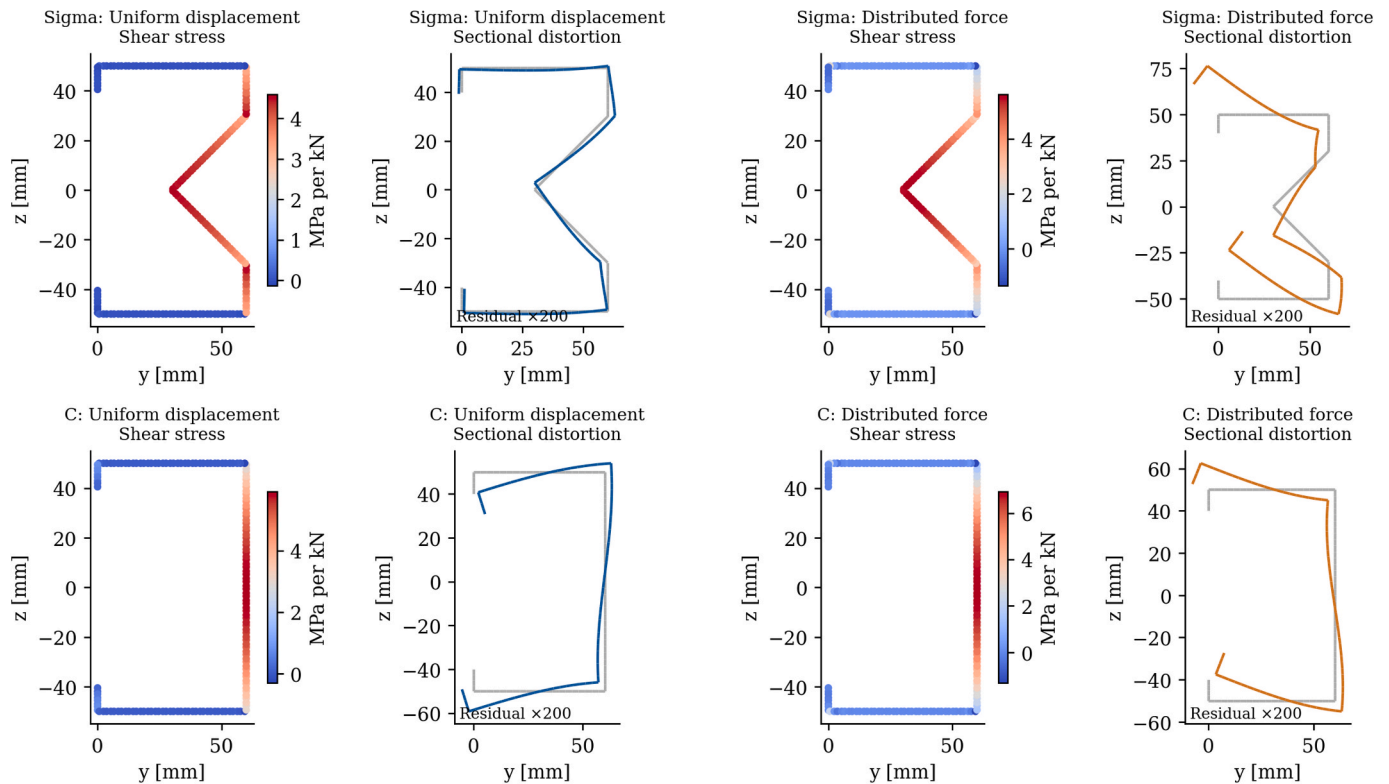


Fig. 14. Sectional shear-stress and residual distortion fields for Sigma and C under UD and DF loading, $L = 350$ mm, $t = 2$ mm, $h_e = 3$ mm. Stress at $x/L = 0.5$ is normalized to 1 kN. Residual distortions at $x/L = 0.4957$ are magnified 200 times; grey lines show undeformed contours.

Table 12

Method differences relative to M3 at $h_e = 2$ mm and observed changes from 5 to 2 mm, $L = 350$ mm and $t = 2$ mm. Refinement changes are not continuum uncertainty bounds.

Section	M1–M3 [%]	M2–M3 [%]	Mesh change M2 [%]	Mesh change M3 [%]
plate	+10.89	−1.22	1.11	0.206
C	−81.93	−2.06	3.38	0.108
CHS	−7.55	−0.97	1.13	0.050

Table 13

Sensitivity to the local shell coefficient at $L = 350$ mm, $t = 2$ mm, and $h_e = 3$ mm under uniform end displacement.

Section	κ_{sh}	M1	M2	M3
Sigma	2/3	0.121432	0.240192	0.244520
Sigma	5/6	0.124203	0.239593	0.244393
Sigma	1	0.126728	0.239045	0.244287
C	2/3	0.072248	0.384210	0.394863
C	5/6	0.072768	0.383469	0.394909
C	1	0.073273	0.382691	0.394933

3.9.5. Global conditioning and force and energy checks

Fig. 15 quantifies Eqs. (17a)–(17b). For the plate at $L/H = 3.5$ and $h_e = 5$ mm, $\beta = 0.947071$ and the local amplification is 17.893. A +1% perturbation of the bending reference changes M1 by −15.05%; a −1% perturbation produces +22.06%. The finite-perturbation expression captures this asymmetry. Conversely, the C-section has modest conditioning amplification but a large method disagreement, demonstrating a separate mechanical limitation.

Table 14 gives the analytical benchmark and equilibrium comparisons. Across all 114 configurations and 39 scan positions, the largest absolute sectional force discrepancy is below $9.76 \times 10^{-8}\%$; at mid-length it is below $8.30 \times 10^{-8}\%$. The largest normalized free-equation residual is 1.85×10^{-11} and the largest relative work discrepancy is 7.33×10^{-10} . The separate shell energy contributions agree with the assembled energy within 8.28×10^{-10} .

Rigid-rotation element checks give normalized residuals below 3×10^{-17} , and a homogeneous uniaxial energy check has relative error below 4×10^{-16} . These tests establish algebraic consistency and selected benchmark accuracy. Equilibrated sectional resultants do not imply exact local stresses or boundary-independent identification.

4. Discussion

The results demonstrate that any quantity reported as a beam-level shear correction factor must be interpreted together with its reference

Table 14

Elementary sectional references and force-equilibrium errors at $L = 350$ mm, $t = 2$ mm, and $h_e = 2$ mm.

Section	Reference	M3	Reference error [%]	Force error [%]
plate	0.833333	0.833663	0.0395	1.01e-08
CHS	0.500000	0.500118	0.0236	1.51e-09

area, loading direction, boundary conditions, identification procedure, and reduced-order beam model. For simple sections in a sufficiently developed beam state, the identified value may approximate an intrinsic sectional shear parameter. For geometrically complex sections or finite members affected by coupled deformation and end-region effects, however, the same numerical procedure may produce a test-dependent effective parameter containing contributions that cannot be attributed uniquely to transverse shear.

4.1. Mechanical meaning of the identified shear parameter

The beam-level definition in Eq. (3) expresses the effective transverse shear stiffness relative to the selected reference area A . In the present study, A is the total material area of the complete thin-walled section. Consequently, k_s is not a material constant and should not be compared directly between different geometries without considering how effectively the complete material area participates in shear transfer.

This effect is illustrated by the I-, C-, Z-, and Omega-sections. Their vertical walls carry much of the applied transverse resultant, while flanges, lips, and other segments also contribute to the total area and complete transverse shear energy. Consequently, web-to-total-area ratios are useful indicators but not exact sectional factors. The numerical values below those ratios reflect both normalization and a nonuniform energy distribution.

The vertical plate provides the simplest verification of this interpretation. Its sectional value agrees closely with the classical rectangular-section result 5/6 and remains near Cowper's Poisson-ratio-dependent solution once the immediate influence of the end regions becomes sufficiently small. This comparison confirms that the shell-based sectional recovery can reproduce a conventional shear correction factor when the cross-sectional response is compatible with the assumptions of the reference beam theory.

The circular section supplies a complementary reference: complete elementary thin-wall shear-flow energy gives $k_s = 1/2$. The regenerated M3 approaches this limit closely for the longer baseline members. Omitting the second transverse energy component produces a substantive inconsistency, exposed by the distinct projected limit of two thirds; finite-length effects do not correct that energy definition.

Global response quantities require a different interpretation. Very low values for the long Z- and C-sections represent residuals of a

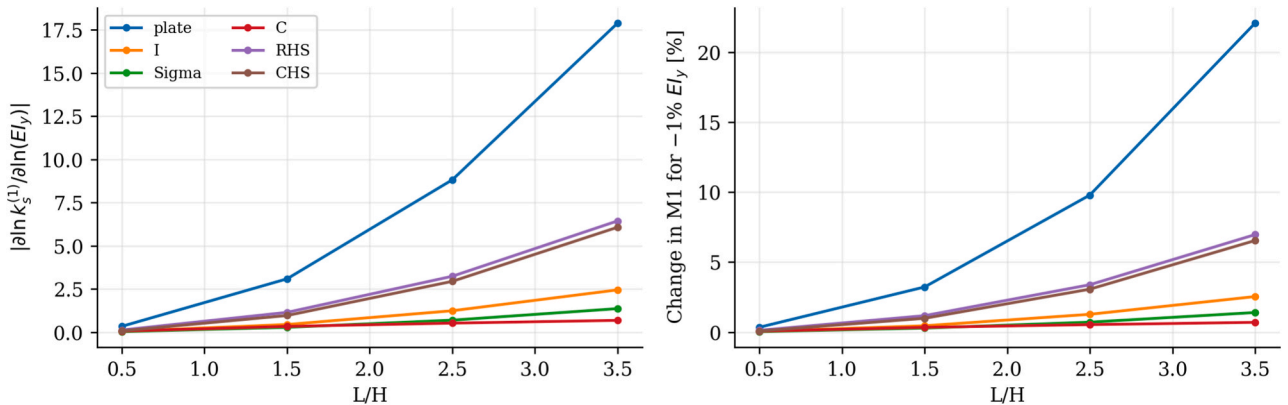


Fig. 15. Local amplification of M1 and its exact change for a −1% perturbation of the bending-reference stiffness, with the solved response held fixed. Cases use $t = 2$ mm and $h_e = 5$ mm.

mechanically incomplete scalar reduction rather than a direct measure of material shear stiffness. Large or otherwise unexpected global values would likewise require a conditioning and kinematic assessment. The circular results remain below unity throughout the present campaign.

4.2. Comparison of the three identification procedures

The three procedures operate at different levels of the shell-to-beam reduction and therefore have different sensitivities.

Method 1 uses the total force–displacement response and subtracts an Euler-Bernoulli bending contribution. Its principal advantage is simplicity: only the resultant reaction force, prescribed displacement, beam length, Young's modulus, and second moment of area are required. For sections whose global response is described adequately by uncoupled bending and transverse shear, the procedure provides a computationally inexpensive effective stiffness.

Its main limitation is that every displacement contribution not represented by the selected Euler-Bernoulli term is assigned implicitly to shear. This includes not only transverse shear but potentially torsion, restrained warping, local wall bending, cross-sectional distortion, and errors caused by using an incomplete flexural representation. The Z-section provides the clearest example. Because the global axes are not principal bending axes and $I_{yz} \neq 0$, the bending compliance cannot be represented by EL_y alone. The bending-compliance discrepancy contributes a term scaling with L^3 to the displacement residual, which is then assigned incorrectly to shear and causes the identified $k_s^{(1)}$ to collapse as member length increases.

A different limitation appears when bending dominates the response of symmetric sections. The shear displacement becomes a small difference between two comparatively large quantities, and deviations in the flexural reference are amplified. The plate example demonstrates this directly: a -1% change in the assumed EL_y produces a $+22.06\%$ change in M1 at $L/H = 3.5$ on the 5 mm mesh. This is a postprocessing sensitivity with the solved response held fixed, not a re-solved material perturbation.

Method 2 integrates both transverse beam shear stress–strain products over the model volume. It avoids analytical subtraction of bending displacement and retains the distinction between local membrane shear and shell transverse shear by transforming both stress and strain before taking their product. This complete transverse energy definition is necessary for consistency with the circular analytical benchmark.

The volume integral contains the developed central region and both disturbed ends. It can retain transverse shear associated with torsion or other coupled response, and is sensitive to raw Gauss-point fields. Local shell bending and twisting energies are reported separately. These distinctions preclude interpretation of the M2–M3 difference as a unique end-region or distortion energy.

Method 3 evaluates the corresponding work-conjugate fields on a selected section using the explicit centre-based axial recovery. It is less mesh-sensitive than M1 in the Sigma study and reproduces the plate and circular references closely in the verified cases. Its interpretation nevertheless requires positional, length, and boundary-condition checks.

The new longitudinal evidence distinguishes spatial stability from transferability. Sigma has an 8.48% central range at $L/H = 3.5$, falling to 0.0167% at 14. C has a small within-member central range but a mid-length value that changes substantially with overall length. The circular field remains close to 0.5004 in the extended models. A universal mid-length sampling rule is therefore not justified.

Methods 2 and 3 retain both transverse energy components but neither supplies a full decomposition of torsion, warping, and distortion. Their agreement supports consistency within the chosen definitions. Their disagreement must be compared with mesh and recovery sensitivity before invoking physical mechanisms, and direct deformation fields do not by themselves supply a modal energy partition.

4.3. Influence of geometry, member slenderness, and discretization

Member slenderness affects the identification procedures through two opposing mechanisms. At low L/H , the clamped and displacement-controlled end regions occupy a substantial part of the member and may overlap. Under these conditions, local plate bending, constrained warping, distortion, and nonuniform shear fields prevent the development of a Saint-Venant-type beam state. This effect is visible in the shortest vertical-plate, Sigma-, C-, Omega-, and circular-section models.

At larger L/H , the relative influence of the two end regions generally decreases for the field-based procedures. This explains why Methods 2 and 3 often approach one another as member length increases. The same increase in length can, however, reduce the robustness of Method 1 because bending grows proportionally to L^3 , whereas the reference shear displacement grows proportionally to L . The shear term therefore becomes a progressively smaller residual within the total compliance.

Cross-sectional topology determines which additional deformation modes are likely to enter this process. Open sections with free edges may develop nonuniform warping and local distortion. Sections whose centroidal axes are not principal axes additionally require a coupled bending description. Symmetric open sections avoid unsymmetric bending but remain susceptible to warping and local wall deformation. Closed sections provide continuous shear-flow paths and greater torsional restraint, producing substantially more stable field-based results. Closure does not, however, eliminate end-region shell behavior or the conditioning problem of global subtraction.

The 2 and 5 mm baseline meshes for plate and I provide mesh-sensitivity information without establishing formal convergence. Additional two-level checks for C and the circular section extend that information. Small changes in M3 do not imply equal accuracy in M1 or M2, because the extraction procedures amplify and average the solved fields differently.

The dedicated Sigma-section study uses four mesh levels and separates discretization effects from differences arising from the mechanical definitions of the procedures. The 1 mm mesh is treated as the finest available numerical reference rather than as an exact continuum solution. The convergence assessment is particularly important for Methods 2 and 3 because both depend directly on recovered stress and strain fields.

Relative to the 1 mm Sigma reference, the 2 mm deviations are 4.854%, 0.694%, and 0.136% for M1, M2, and M3. Only the two field-based procedures satisfy the 1% criterion. The finest M1–M3 separation clearly exceeds the final mesh change, whereas the smaller M2–M3 separation is comparable to M2 sensitivity. The latter cannot confidently be interpreted as a purely physical effect.

Shear locking cannot be excluded solely from agreement between two mesh levels. The adopted fully integrated displacement-based Mindlin–Reissner element is not intrinsically locking-free. Its numerical behavior must therefore be evaluated using the multi-level convergence results, the wall-thickness study, and comparison with the analytical benchmarks. Accordingly, no discretization is described as mesh-independent unless this conclusion is supported quantitatively.

4.4. Implications for dimensional reduction and composite structures

The shell model and the reduced beam model serve different computational purposes. The shell representation resolves the topology and plate-like mechanics of the thin walls and is used as an offline identification model. Once a mechanically meaningful effective sectional stiffness has been obtained, it can be transferred to a computationally efficient beam formulation for repeated simulations, large structural systems, optimization, uncertainty analysis, or multiscale modelling.

A scalar shear correction factor is sufficient only when the relevant structural response can be represented adequately by uncoupled

bending and transverse shear in one loading plane. The Z-section results show that even the flexural contribution may require a coupled sectional inertia matrix. For open profiles exhibiting important torsion and warping, a more complete reduced model should include bending–torsion coupling, warping degrees of freedom, or a coupled sectional stiffness matrix rather than a single scalar k_s .

This conclusion is particularly relevant to composite structures. Laminated, sandwich, and multi-material thin-walled members may exhibit material anisotropy, extension–shear coupling, bending–twisting coupling, and nonuniform through-thickness behavior. The breadth of nonlinear and coupled responses in lightweight composite structures is discussed in [39], while [40] provides a representative multiphysics example involving a composite cylindrical shell. Extension of the present framework to such sections is feasible at the shell level through appropriate anisotropic and layerwise constitutive descriptions. The reduced-order identification, however, should generally target a coupled sectional stiffness matrix rather than assume that one independent shear correction factor is sufficient.

The isotropic steel sections used in the present study provide controlled geometric benchmarks. Related studies concerning layered corrugated board, porous heterogeneous media, multiwall corrugated structures, non-homogeneous sections with embedded cores, and corrugated sandwich plates are reported in [44–48]. Together, these applications demonstrate that analogous shear-stiffness identification problems arise in heterogeneous, layered, and multi-material systems. In the present benchmarks, the effects of topology, member slenderness, boundary conditions, discretization, and extraction procedure can be examined before constitutive anisotropy and layer coupling are introduced. The results should therefore be interpreted as validation of the numerical reduction strategy for isotropic benchmark sections, not as direct validation for laminated composites.

Classical and recent formulations for laminated, functionally graded, anisotropic, and heterogeneous structural members [26,27,35,36,44–48] further demonstrate that model fidelity must be matched to the deformation mechanisms retained in the reduced theory. The present findings complement these developments by showing that equivalent-stiffness identification is reliable only when the detailed shell response and the assumed beam kinematics represent compatible sets of deformation modes.

4.5. Limitations and future developments

The study is restricted to prismatic, homogeneous, isotropic, linear-elastic sections loaded in one transverse direction and one fully integrated shell-element family. The baseline displacement-controlled arrangement is supplemented by alternative loading for two open sections, longitudinal scans, and selected length extensions. These checks extend the evidence but do not establish universal validity for all sections and loading arrangements.

The models do not include material plasticity, geometric nonlinearity, residual stresses, local buckling, contact, joints, or manufacturing imperfections. The identified parameters therefore apply only to the pre-buckling linear-elastic response. For sections susceptible to local instability, a constant linear shear stiffness may cease to represent the member response once wall buckling or yielding begins. Classical shell-stability formulations provide the broader context for this limitation [33].

The study does not provide a complete energetic separation of transverse shear, Saint-Venant torsion, restrained warping, distortion, and local plate bending. Both transformed transverse shear products are retained, while shell bending energy remains separate. A coupled sectional stiffness matrix and independent generalized torsional and warping measures would be required for a fuller reduction, as suggested by enriched formulations [28,38].

Distributed-force tests for Sigma and C quantify sensitivity to the adopted end arrangement. They change the load distribution and

kinematic restraint together and are not loading-through-shear-centre tests or isolated releases of one mode. Further comparisons with controlled end rotations, rigid diaphragms, and independent sectional boundary conditions remain useful.

The analytical verification is presently limited to the vertical plate and circular thin-wall reference. An independent comparison with three-dimensional solid finite elements, higher-order shell models, variational asymptotic sectional analysis, or experimentally measured global and sectional responses would provide a stronger validation hierarchy.

Four-level refinement is restricted to one complex open profile. The supplementary two-level checks for C and the circular section do not establish general convergence for every geometry or very thin wall. The fully integrated element may exhibit locking or parasitic strains, and a small force-equilibrium residual cannot exclude pointwise field errors.

Extension to composite and multi-material sections will require anisotropic shell constitutive laws, layerwise stress recovery, and identification of coupled stiffness terms. Tapered, stepped, curved, and otherwise non-prismatic members will additionally require spatially varying effective properties rather than one constant sectional parameter.

Within these limitations, the present framework provides a systematic diagnostic procedure. Agreement between global, volume-integrated, and sectional estimates supports the use of a scalar effective transverse shear stiffness. Persistent disagreement indicates that the assumed beam reduction is incomplete, that the benchmark remains affected by end-region or discretization effects, or that the cross-section requires a coupled reduced-order representation.

5. Conclusions

This study developed a shell-based finite element framework for identifying effective transverse shear stiffness in thin-walled sections and assessing transfer to a reduced Timoshenko-type beam model. The framework was demonstrated using isotropic steel benchmarks and three procedures: global compliance, volume integration of complete transverse beam shear energy, and recovered sectional shear fields. The revised campaign includes 114 distinct executed configurations and explicit checks of recovery, equilibrium, end loading, and numerical conditioning.

The principal conclusions are as follows:

1. The identified parameter is not universally an intrinsic shear correction factor. Its physical meaning depends on the reference area, loading direction, boundary conditions, member slenderness, cross-sectional topology, and deformation modes represented by the reduced beam theory. For sections exhibiting coupled bending, torsion, restrained warping, or local distortion, the result should be interpreted as an effective parameter of the prescribed benchmark configuration.
2. The sectional procedure gives the closest agreement with the elementary plate and circular references in the verified cases, with deviations of 0.040% and 0.024% on the investigated 2 mm meshes. Its physical interpretation still depends on field development and benchmark compatibility. Both transverse shear-energy components are required; a loading-direction projection alone gives the wrong elementary circular limit.
3. The volume-integrated procedure avoids subtraction of analytical bending energy but retains the full member domain and raw Gauss-point recovery. Its difference from M3 can therefore combine spatial and numerical effects. It should not be interpreted directly as an independently isolated torsion, warping, or distortion contribution.
4. The global compliance procedure is reliable only when the shell response is compatible with the assumed bending-plus-shear beam model. Every displacement contribution not represented by the Euler–Bernoulli bending term is assigned implicitly to

shear. For the Z-section, the use of EI_y alone is insufficient because the nonzero product of inertia produces unsymmetric bending. For long, bending-dominated members, even symmetric and closed sections may yield poorly conditioned global estimates because the shear displacement is obtained as a small residual between two comparatively large quantities.

5. Unexpected global values should be treated as diagnostic indicators. In the plate example, a -1% change in the assumed bending reference changes M1 by $+22.06\%$. For some open sections, large global-sectional differences persist even when this conditioning amplification is small, demonstrating a separate limitation associated with reduced-model kinematics.
6. Cross-sectional closure improves consistency in the investigated examples but is not a sufficient validity criterion. The circular complete-energy benchmark is especially useful for detecting an omitted energy component. Agreement and force equilibrium alone do not establish pointwise accuracy or independence from end loading.
7. Member length affects spatial field development and global subtraction differently. Sigma does not exhibit a stable central field at the original $L/H = 3.5$, but its central range falls below 0.017% at 14. For C, a locally flat field can coexist with substantial length dependence. Mid-length selection therefore requires evidence rather than geometric distance alone.
8. The executed Sigma refinement study gives 2-to-1 mm changes of 4.854% , 0.694% , and 0.136% for M1, M2, and M3. The finest result is a numerical reference, not a continuum solution. The smaller M2–M3 difference remains comparable to numerical sensitivity and is not assigned wholly to physical coupling.
9. A scalar shear parameter is sufficient only when the reduced beam kinematics retain the deformation mechanisms present in the shell model. Sections with important bending–torsion coupling, warping, or distortion require a more complete reduced representation, potentially including a coupled sectional stiffness matrix and additional generalized deformation modes.
10. The present isotropic steel sections constitute benchmark cases rather than direct composite validation. Extension to laminated, anisotropic, or multi-material sections will require corresponding shell constitutive matrices, layerwise field recovery, and identification of coupled sectional stiffness terms rather than one independent scalar correction factor.

The framework is both an identification and a diagnostic tool. Its use requires a compatible energy definition, transparent recovery, numerical checks, and evidence that the finite-member configuration represents the intended beam response. Persistent disagreement calls for a more complete reduction or additional verification; isolated agreement does not justify an intrinsic sectional constant.

CRedit authorship contribution statement

Tomasz Garbowski: Writing – review & editing, Writing – original draft, Visualization, Supervision, Software, Methodology, Investigation, Funding acquisition, Formal analysis, Conceptualization. **Tomasz Gajewski:** Writing – review & editing, Writing – original draft, Visualization, Supervision, Software, Methodology, Investigation, Funding acquisition, Formal analysis, Conceptualization. **Anna Szymczak-Graczyk:** Writing – review & editing, Validation, Resources, Project administration, Investigation, Formal analysis, Data curation.

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Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Data availability

The data that supports the findings of this study are available from the corresponding author upon reasonable request.

References

- [1] M.A. Neto, A. Amaro, R. Luis, J. Cirne, R. Leal, in: *Engineering computation of structures: The finite element method*, Springer, 2015, <https://doi.org/10.1007/978-3-319-17710-6>.
- [2] D. Thang, An improved method of calculating beam deformation considering transverse shear strain, *Int. J. GEOMATE* 20 (83) (2021) 39–46, <https://doi.org/10.21660/2021.83.j2059>.
- [3] D. Biro, A one-dimensional introduction to the finite element method, [in:] *Computational Electromagnetics for RF and Microwave Engineering*, 2nd ed., Cambridge University Press, 2010, 331–366.
- [4] Z. Kączkowski, *Płyty: Obliczenia Statyczne*, Arkady, Warsaw, 2000.
- [5] S. Timoshenko, S. Woinowsky-Krieger, *Theory of Plates and Shells*, 2nd ed., McGraw-Hill, New York, 1959.
- [6] L.H. Donnell, *Beams, Plates and Shells*, McGraw-Hill, New York, 1976.
- [7] A. Nishida, Effect of Timoshenko coefficient in wave propagation analysis of a three-dimensional frame structure, *Theor. Appl. Mech. Jpn* 56 (2008) 57–67, <https://doi.org/10.11345/nctam.56.57>.
- [8] R. Li, Research on application of finite element method in static analysis, *IOP Conf. Ser.: Earth Environ. Sci.* 1802 (4) (2021) 042008.
- [9] S.P. Timoshenko, On the correction for shear of the differential equation for transverse vibrations of prismatic bars, *Phil. Mag.* 41 (245) (1921) 744–746.
- [10] S.P. Timoshenko, On the transverse vibrations of bars of uniform cross-section, *Phil. Mag.* 43 (253) (1922) 125–131.
- [11] G.R. Cowper, The shear coefficient in Timoshenko's beam theory, *J. Appl. Mech.* 33 (2) (1966) 335–340, <https://doi.org/10.1115/1.3625046>.
- [12] J.R. Hutchinson, Shear coefficients for Timoshenko beam theory, *J. Appl. Mech.* 68 (1) (2001) 87–92, <https://doi.org/10.1115/1.1349417>.
- [13] J. Lewandowski, K. Mysłęcki, Modified Hu-Washizu variational principle as a general basis for FEM plasticity equations, [in:] *Advances in Mechanics: Theoretical, Computational and Interdisciplinary Issues*, CRC Press, 2016, 433–444.
- [14] V.P. Trubitsyn, A.P. Trubitsyn, Comparative analysis and unified derivation of Reissner's equations for 2D bending of thick plates and Timoshenko's equations for bending of beams, *Izv. Phys. Solid Earth* 60 (1) (2024) 110–122, <https://doi.org/10.1134/S1069351324700356>.
- [15] S. Rendek, I. Baláz, Distortion of thin-walled beams, *Thin-Walled Struct.* 42 (2) (2004) 255–277, [https://doi.org/10.1016/S0263-8231\(03\)00059-4](https://doi.org/10.1016/S0263-8231(03)00059-4).
- [16] C. Liu, Forced vibration analysis of thin-walled beams under harmonic moving loads, *J. Comput. Methods Sci. Eng.* 25 (5) (2025) 1709–1724, <https://doi.org/10.1177/14727978251364449>.
- [17] F. Gruttmann, W. Wagner, Shear correction factors in Timoshenko's beam theory for arbitrary shaped cross-sections, *Comput. Mech.* 27 (3) (2001) 199–207, <https://doi.org/10.1007/s004660100239>.
- [18] V.Z. Vlasov, *Thin-Walled Elastic Beams*, Israel Program for Scientific Translations, Jerusalem, 1961.
- [19] K.J. Bathe, *Finite Element Procedures*, 2nd ed., Klaus-Jürgen Bathe, Watertown, 2014.
- [20] O.C. Zienkiewicz, R.L. Taylor, J.Z. Zhu, *The Finite Element Method: its Basis and Fundamentals*, 7th ed., Butterworth-Heinemann, Oxford, 2013.
- [21] W.D. Pilkey, *Analysis and Design of Elastic Beams: Computational Methods*, John Wiley & Sons, New York, 2002.
- [22] R.D. Cook, D.S. Malkus, M.E. Plesha, R.J. Witt, *Concepts and Applications of Finite Element Analysis*, 4th ed., John Wiley & Sons, New York, 2001.
- [23] X.F. Li, B.L. Wang, Vibrational modes of Timoshenko beams at small scales, *Appl. Phys. Lett.* 94 (2009) 101903, <https://doi.org/10.1063/1.3094130>.
- [24] R.K. Kovvali, D.H. Hodges, Verification of the Variational Asymptotic Beam Section (VABS) analysis for initially curved and twisted beams, [in:] *48th AIAA/ASME/ASCE/AHS/ASC Structures Conference*, 2007, 1852. doi:10.2514/6.2007-1852.
- [25] S. Vlase, M. Marin, A. Öchsner, O. El Moutea, Equivalent analytical formulation-based multibody elastic system analysis using one-dimensional finite elements, *Contin. Mech. Thermodyn.* 36 (5) (2024) 2079–2096, <https://doi.org/10.1007/s00161-023-01270-4>.
- [26] J.N. Reddy, *Mechanics of Laminated Composite Plates and Shells: Theory and Analysis*, 2nd ed., CRC Press, Boca Raton, 2004.

- [27] W.Y. Jung, S.C. Han, W.T. Park, Four-variable refined plate theory for forced-vibration analysis of sigmoid functionally graded plates on elastic foundation, *Int. J. Mech. Sci.* 111 (2016) 15–26, <https://doi.org/10.1016/j.ijmecsci.2016.03.001>.
- [28] Y.Z. Liu, Y.B. Yang, X.H. Liu, D.Z. Guo, H. Xu, A novel beam element with nine DOFs per node for resolving the warping-distortion compatibility in analysis of frames and curved beams made of I-sections, *Thin-Walled Struct.* 194 (2024) 111314, <https://doi.org/10.1016/j.tws.2023.111314>.
- [29] E. Esmailzadeh, D. Younesian, H. Askari, Integral based methods, [in:] *Solid Mechanics and its Applications*, Springer, 2019, 165–221. doi:10.1007/978-94-024-1542-1_6.
- [30] S.E. Mousavi, N. Sukumar, Generalized Gaussian quadrature rules for discontinuities and crack singularities in the extended finite element method, *Comput. Methods Appl. Mech. Eng.* 199 (45–46) (2010) 2974–2986, <https://doi.org/10.1016/j.cma.2010.06.031>.
- [31] A.Y. Chirkov, Some applications of the mixed finite-element method to the solution of problems in solid mechanics, *Cybern. Syst. Anal.* 48 (3) (2012) 364–370, <https://doi.org/10.1007/s10559-012-9454-3>.
- [32] S.B. Dong, C. Alpdogan, E. Taciroglu, Much ado about shear correction factors in Timoshenko beam theory, *Int. J. Solids Struct.* 47 (13) (2010) 1651–1665, <https://doi.org/10.1016/j.ijsolstr.2010.02.018>.
- [33] L.P. Kollár, E. Dulácska, *Buckling of Shells for Engineers*, John Wiley & Sons, Chichester, 1984.
- [34] W. Yu, D.H. Hodges, Variational asymptotic beam section analysis—An updated implementation, *Int. J. Eng. Sci.* 159 (2021) 103433, <https://doi.org/10.1016/j.ijengsci.2020.103433>.
- [35] E. Carrera, A. Pagani, M. Petrolo, E. Zappino, Recent developments on refined theories for beams with applications, *Mech. Eng. Rev.* 2 (2) (2015) 14–00298, <https://doi.org/10.1299/mer.14-00298>.
- [36] S. Puchegger, D. Loidl, K. Kromp, H. Peterlik, Hutchinson's shear coefficient for anisotropic beams, *J. Sound Vib.* 262 (5) (2003) 1075–1086, [https://doi.org/10.1016/S0022-460X\(02\)01532-8](https://doi.org/10.1016/S0022-460X(02)01532-8).
- [37] T. Kaneko, On Timoshenko's correction for shear in vibrating beams, *J. Phys. D Appl. Phys.* 8 (16) (1975) 1927–1936.
- [38] J. Murín, V. Kutíš, An effective finite element for torsion of constant cross-sections including warping, *Adv. Eng. Softw.* 39 (1) (2008) 9–16, <https://doi.org/10.1016/j.advengsoft.2007.03.002>.
- [39] Y. Liu, D. Vella, Z. Qin, F. Chu, M. Amabili, A review of nonlinear dynamics and vibration in lightweight composite structures: recent advances and challenges, *Eng. Struct.* 360 (2026) 122835, <https://doi.org/10.1016/j.engstruct.2026.122835>.
- [40] Y. Liu, Z. Qin, F. Chu, Investigation of magneto-electro-thermo-mechanical loads on nonlinear forced vibrations of composite cylindrical shells, *Commun. Nonlinear Sci. Numer. Simul.* 107 (2022) 106146, <https://doi.org/10.1016/j.cnsns.2021.106146>.
- [41] P. Wang, Y. Liu, R. Zhu, L. Qin, J. Deng, Z. Yang, Z. Qin, F. Chu, Dynamic analysis of vibro-impact energy harvester with acoustic black hole, *Int. J. Mech. Sci.* 291–292 (2025) 110193, <https://doi.org/10.1016/j.ijmecsci.2025.110193>.
- [42] I. Elishakoff, The tale of shear coefficients in Timoshenko–Ehrenfest beam theory: 130 years of progress, *J. Sound Vib.* 473 (2020) 115226, <https://doi.org/10.1016/j.jsv.2020.115226>.
- [43] H. Altenbach, *Theories of Plates and Shells*, Springer, Berlin, 2004.
- [44] T. Garbowski, J. Graczyk, Shear correction factor in corrugated board layered plates, *Pol. Paper Rev.* 81 (12) (2025) 686–690, <https://doi.org/10.15199/54.2025.12.3>.
- [45] J. Graczyk, T. Gajewski, T. Garbowski, Shear correction factor for porous eco-materials: mechanical characterization of a heterogeneous medium, *Buildings* 16 (4) (2026) 0688, <https://doi.org/10.3390/buildings16040688>.
- [46] J. Graczyk, J. Tworzydło, T. Garbowski, Parametric sensitivity of shear correction factors for multiwall corrugated structures, *Materials* 19 (2026) 863, <https://doi.org/10.3390/ma19050863>.
- [47] A. Szymczak-Graczyk, Z. Guri, I. Canaj, T. Garbowski, Generalized shear correction factor for non-homogeneous beam cross-sections with an embedded steel core, *Struct. Durab. Health Monit.* (2026) 26538, <https://doi.org/10.32604/sdhm.2026.080104>.
- [48] T. Garbowski, A. Szymczak-Graczyk, Z. Pozorski, An energy-consistent framework for shear correction in corrugated sandwich plates with soft cores, *Compos. B Eng.* 324 (2026) 113892, <https://doi.org/10.1016/j.compositesb.2026.113892>.
- [49] T.J.R. Hughes, *The Finite Element Method: Linear Static and Dynamic Finite Element Analysis*, Dover Publications, Mineola, NY, 2000.
- [50] Y.-Y. Ding, L.-H. Bai, W.-F. Chen, Y.-P. Liu, S.-L. Chan, A generalized method for shear correction factors of arbitrary thin-walled sections, *Adv. Steel Constr.* 19 (2023) 223–241, <https://doi.org/10.18057/IJASC.2023.19.3.4>.
- [51] Z. Wu, Y. Qiu, H. Jia, H. Zhang, H. Ye, Y. Zheng, Automatic data-driven modeling of plate and beam elements based on the first principles of elasticity, *Thin-Walled Struct.* 209 (2025) 112939, <https://doi.org/10.1016/j.tws.2025.112939>.
- [52] A. Hellers, M. Reichle, S. Klinkel, A polygonal Reissner–Mindlin plate element based on the scaled boundary finite element method, *Comput. Mech.* 77 (2026) 2071–2095, <https://doi.org/10.1007/s00466-025-02742-x>.
- [53] H.G. Lee, D.-K. Choi, D. Kwon, S. Park, Refined shear correction factors for composite-layered FE shell elements to enhance the accuracy of their modal analysis results, *Compos. Struct.* 377 (2026) 119851, <https://doi.org/10.1016/j.compstruct.2025.119851>.