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DAMAGE DETECTION THROUGH WAVELET TRANSFORM AND INVERSE ANALYSIS

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Abstract. Detection, localization and estimation of details of concentrated defects hidden in structural elements as an important part of structural health monitoring is considered here. In this work the effectiveness of discrete wavelet transform combined with inverse analysis is also discussed. The efficiency of the method is studied particularly when applied to eigenmodes of a cantilever steel beam expressed in amplitudes of vertical displacements, velocities, accelerations or strains. The structural response signal measured in discrete points is transformed using wavelet decomposition which clearly improve identifiability of damaged structure. Authors use a parametrized finite element model which mimic the real structure and by changing control parameters embedded in the numerical model minimize the discrepancy between the wavelet representation of both 'real' and numerically computed measurable quantities. For the discrepancy function minimization (within least-square framework) the deterministic, iterative Trust Region Algorithm is used. Also another technique which is applied to minimize the objective function within a frame of global minimization techniques i.e. Genetic Algorithm is tested and checked here.

1 INTRODUCTION

The problem of damage detection belongs to a wide class of identification problems, where unknown parameters of a structure are determined from experimental tests. It is connected with structural health monitoring and safety assessment. The damage can have different forms such as cracks, voids, inclusions or delamination, often found in composites. Localized damage is extremely dangerous because it can initiate progressive failure of a whole structure.

Among a large number of non-destructive testing X-rays, vibration, acoustic emission, heat transfer, magnetic field , eddy current or ultrasonic methods (see e.g. [1], [2], [3], [4],

[5], [6], [7]) also wavelet transform combined with inverse analysis can be used. Because even small and local damage leads to stiffness reduction, increase of damping and decrease of a natural frequency of the structure, damage detection methods based on analysis of structural dynamic response can be easily applied to identify the presence of damage [8]. It happens that the experiments limited to measurement of eigenfrequences are insufficient, since the global dynamic response is rather insensitive to damage localized on a small area, therefore the localization and severity of defect is not easy to identify. On the other hand methods based on local inspection or heat generation are capable to find damage position, form and/or magnitude but have small range of applicability.

For structural health monitoring different types of response, namely eigenfrequencies/eigenmodes, displacements, velocities and accelerations can be monitored. For this purpose modern scanning laser vibrometer for non-contact measurement are often applied. Vibrometer is capable of gathering vibration data in all three-dimensional coordinate system and have an extended range of ultra high vibration frequencies up to 600 MHz *www.ects.pl*.

The most fundamental challenge is the fact that damage is typically local phenomenon and may not significantly influence the global response of structures. Therefore the method which enables to extract the desired detailed information from a numerous data representing the global response of a damaged structure called Wavelet Transformation (WT) is proposed. Signal decomposition using WT allows to detect and localize the damage because wavelets demonstrate strong disturbance in a place where some defect is present. There are many wavelet functions e.g. Haar wavelet, Symlet, Coiflet, Meyer, Mexican hat or Morlet and new ones are constantly developed. It follows from the experience (see [9]) that in the class of considered problems the most effective appeared Daubechies wavelet of 4th order with two vanishing moments [10]. Estimation of the magnitude of the damage can be done by making use of e.g. Lipschitz exponent [11]. However, data processing of the structural response signal using CWT or DWT has appeared to be rather ineffective in identification of the type or shape of a defect. Therefore, some alternative method which provides a more precise damage identification is needed. In the literature a few attempts can be found, e.g. a combination of WT and artificial neural networks [12] or with inverse analysis [13].

The inverse analysis provides an important tool if one would like to characterize a bigger number of damage parameters in the locally deteriorated elements of the structures. Such technique uses, besides the experimentally obtained data (here the wavelet representations of the experimental measurements), also their numerical counterparts obtained from the computer test simulation. In the inverse analysis a variety of different minimization techniques can be employed for the discrepancy minimization. The discrepancy between experimental and numerical measurable quantities, called the objective function, is usually minimized in the frame of least square approach [14]. In the literature many authors solve an inverse problems by making use of iterative minimization gradient based algorithms (e.g. [15]), based on soft computing methods (e.g. [16]), etc. The inverse analysis using any minimization algorithm, searches for a set of embedded (usually unknown or uncertain) structural or constitutive parameters (not easily accessible from the experiment) by making use of indirect measurement. Such approach was successfully used by many researchers in various fields (e.g. [17]). However, the application of WT in its discrete form together with inverse analysis for structural diagnosis is still an open and unsolved problem.

2 BASIS OF WAVELET THEORY

Wavelets are functions that satisfying certain mathematical requirements can be used to represent data or other functions. Nevertheless this concept is not new. In the early 1800's Joseph Fourier, French mathematician, discovered that using superposition of sines and cosines he could represent other functions. Fourier transform is a perfect tool for analyzing the stationary signals representing them in frequency domain. Wavelets have advantages over it in situations when the signal contains discontinuities, spikes or sharp edges. In wavelet transform the data are cut into different frequency components and then each component is analyzed with resolution matched to its scale. It reduces the effects of the Heisenberg uncertainty principle [18], which in this case means the inability of precise signal analysis in time domain and frequency domain at the same time.

Fourier transform is a basic tool for harmonic analysis and signal processing. It decomposes a function/signal into sinusoids of different frequencies. The transformation is reversible and lossless and the function can be reconstructed from its transform. Fourier transform is defined over the space $\mathbf{L}^2(\mathbb{R})$ of square-integrable functions.

Fourier transform represents a signal through a linear combination of basis function and is defined as:

$$F(\omega) = \int_{-\infty}^{\infty} f(t) \cdot e^{-i\omega t} dt = \left\langle f(t),\, e^{i\omega t} \right\rangle, \qquad f \in \mathbf{L}^2(\mathbb{R}), \tag{1}$$

where i is the imaginary unit ($i^2 = -1$), ω - circular frequency $[rad/s]$ and t is time. The inner product of (1) can be written in the form:

$$\langle f,\, g\rangle = \int_{t \in R} f(t) \cdot \bar{g}(t)\, dt, \tag{2}$$

where $\bar{g}(t)$ is the complex conjugate of $g(t)$ function.

The Fourier coefficient $F(\omega)$ is obtained by multiplying function f and sinusoidal wave $e^{i\omega t}$. As the $e^{i\omega t}$ covers the entire real axis, the value of $F(\omega)$ depends on the values of $f(t)$ for all $t \in (\mathbb{R})$. It is therefore difficult to analyze any local properties of f on the basis of $F(\omega)$. Such analysis requires the decomposition of the signal using the set of functions well localized in time and frequency. Wavelet transformation is well suited for this purpose.

It is considered that $\psi\ (t) \in \mathbf{L}^2(\mathbb{R})$ is a wavelet (mother) function if it satisfies admissibility condition:

$$\int_0^{\infty} \frac{|\Psi(\omega)|^2}{\omega} d\omega < \infty, \tag{3}$$

where $\Psi(\omega)$ is Fourier transform of function $\psi(t)$.
Average value of wavelet function is equal to zero, it means that the wavelet integral over real axis disappears:

$$\int_{-\infty}^{\infty} \psi(t)\, dt = 0. \tag{4}$$

In wavelet transform there is only one wavelet (mother) function. For signal decomposition copies of wavelet, which are called wavelet family, are used. They are obtained by scaling and translating ψ according to formula:

$$\psi_{a,b} = \frac{1}{\sqrt{|a|}} \psi \left(\frac{t-b}{a} \right), \tag{5}$$

where the variable t denotes time or space coordinate, a is the scale parameter and b indicates the wavelet translation in time/space domain; $a,\ b \in (\mathbb{R})$; $a \neq 0$. The scale factor $|a|^{-1/2}$ is a normalization coefficient which ensures constant wavelet energy regardless of the scale. This means that $\|\psi_{a,b}\| = \|\psi\| = 1$ [19].

Continuous wavelet transform of given function $f(t)$ is obtained by integration the product of the signal function and the wavelet functions [20]:

$$W\ f\ (a,b) = \frac{1}{\sqrt{|a|}} \int_{-\infty}^{\infty} f\ (t) \cdot \bar{\psi} \left(\frac{t-b}{a} \right) dt = \langle\ f\ (t),\ \psi_{a,b} \rangle\ , \qquad f \in \mathbf{L}^2(\mathbb{R}). \tag{6}$$

The inner product of (6) can be written in the form:

$$\langle\ f\ (t),\ \psi_{a,b} \rangle = \int_{t \in R} f\ (t) \cdot \bar{\psi}_{a,b}\, dt, \tag{7}$$

where $\bar{\psi}_{a,b}$ is the complex conjugate of $\psi(t)$ wavelet.

On the basis of formulas (1), (2), (6) and (7) it can be concluded that the wavelet transform is a transformation similar to the Fourier transform. Both of them are based on the use of the product of a signal $f\ (t)$ and the remaining part, called the kernel of the transform. The main difference is that the kernel in Fourier transform are sinusoidal functions (periodic, representing one frequency) and in wavelet transform the kernel is wavelet function which satisfies conditions (3) and (4). Next dissimilarity is that wavelet functions are localized in space. Fourier sinusoidal functions are not.

An important role in applications plays a dyadic wavelet transformation. Substituting $a = 1/2^j$ and $b = k/2^j$, $k,\ j \in \mathcal{C}$ in the (5) a wavelet family is obtained:

$$\psi_{j,k}(t) = 2^{j/2}\psi\left(2^j t - k\right), \tag{8}$$

where $j = 0, ..., J-1$ is scale parameter, $k = 0, ..., 2^j - 1$ translation parameter and J is the maximum level of transformation.
Discrete wavelet transform is defined as:

$$W\,\psi_{j,k} = 2^{j/2}\int_{-\infty}^{\infty} f\,(t)\cdot\bar{\psi}\left(2^j t - k\right)dt = \langle\, f\,(t),\ \psi_{j,k}\rangle \tag{9}$$

and wavelet coefficients are given by:

$$d_{j,k} = \langle\, f\,(t),\ \psi_{j,k}\rangle\,. \tag{10}$$

A linear combination of wavelet functions $\psi_{j,k}$ and wavelet coefficients $d_{j,k}$ allows to represent a discrete signal (the number of data is equal to 2^J) in the form:

$$f(t) = \sum_{j=0}^{J-1} d_{j,k}\,\psi_{j,k}(t) \tag{11}$$

Multi-level signal representation is possible thanks to multi-resolution analysis (MRA) (see [21]), closely connected with wavelet transform. For multi-resolution signal analysis a scaling wavelet $\varphi(t)$ (father) is required:

$$\varphi_{j,k}(t) = 2^{j/2}\varphi\left(2^j t - k\right). \tag{12}$$

Discrete signal $f(t)$ can be approximate using wavelet $\psi(t)$ and scaling $\varphi(t)$ functions according to:

$$f(t) = \sum_{k=-\infty}^{\infty} a_{j,k}\cdot\varphi_{j,k}(t) + \sum_{k=-\infty}^{\infty}\sum_{j=0}^{\infty} d_{j,k}\cdot\psi_{j,k}(t), \tag{13}$$

where $a_{j,k}$ are scaling function coefficients derived from the formula:

$$a_{j,k} = \langle\, f\,(t),\ \varphi_{j,k}\rangle\,. \tag{14}$$

A wavelet function has a band-like spectrum, so the coefficients $d_{j,k}$ have high frequencies information (details). Coefficients $a_{j,k}$ have low-pass information with a constant component which is called signal approximation.

Multi-resolution analysis of discrete signal can be expressed in Mallat's algorithm:

$$f_J = S_J + D_J + ... + D_n + ... + D_1, \qquad n = J - j \tag{15}$$

where S_J is a smooth signal representation, D_n and are S_n are details and rough parts of a signal, j is the level of decomposition and J level of MRA. The idea of multiresolution analysis using Mallat pyramid is presented in Fig.1.

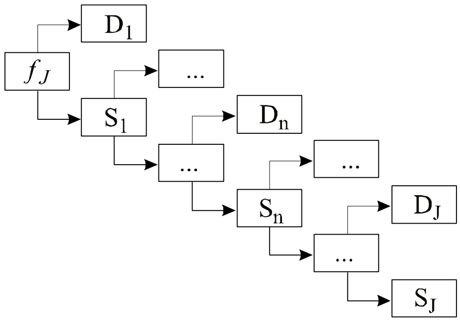

Figure 1: Multiresolution analysis of discrete signal

3 PROBLEM FORMULATION

Studies on the identification of defects reported in the literature are most often related to the beams, frames or plates (see e.g. [22]). They are in fact structural elements, which are most common in engineering practice. However, the specific type of the structure does not make any difference, provided that the response signal for any action (not necessarily defined) can be received. The main task of this study is to detect localization of damage in the structure, if such damage exists. Moreover the position, type, shape and severity of defect should be found. A cantilever beam made of steel, with Young's modulus $E = 200$ GPa and mass density $\rho = 7850\text{kg/m}^3$ is considered. The length of the beam is 0.96 m and rectangular cross section has dimensions 4×8 cm. Damage in beam is modeled as local stiffness reduction, obtained by reducing the height of cross-section or the value of Young's modulus. Authors use a parametrized Finite Element (FE) model which mimic the real structure subjected to dynamic mechanical excitation. All control parameters gathered in the vector $\mathbf{x}$ are embedded in the numerical model; by changing them one can minimize the discrepancy between the wavelet representation of both 'real' and numerically computed measurable quantities. Here the 'real' experiment is substituted with a numerical one, called here pseudo-experiment, in which all parameters (i.e. damage localization, type or shape of damage, number of monitored points, etc.) are known (Fig. 2a). By different initialization of the vector $\mathbf{x}$ in the numerical model (which is different from the pseudo-experimental one) (Fig. 2b), and by comparing the converged values of the sought parameters to those parameters used for pseudo-experimental data generation, one can check the robustness of the proposed method.

The effectiveness of the method was studied when applied to eigenmodes expressed in amplitudes of vertical displacements, velocities, accelerations or strains. The structural response of this kind is a discrete signal measured in points uniformly distributed along the span of a beam and transformed using WT. The response of undamaged structure in such case is unnecessary.

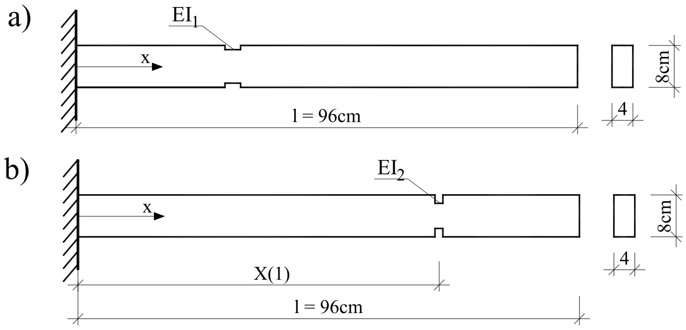

Figure 2: Steel cantilever beam: a) real damaged structure (experiment), b) computer model with variable defect parameters.

4 PROBLEM SOLUTION AND RESULTS

Among a large group of optimization algorithms in frame of nonlinear least square problems, the Gauss-Newton (GN) or Levenberg-Marquardt (LM) (see e.g. [23, 24] for more details) are the most efficient and often implemented into the inverse procedure. Here, however, another powerful algorithm is programmed and used for objective function minimization, namely trust region algorithm (TRA). TRA uses a simple idea, similar to the one in LM algorithm, where the new step is performed in the direction which combines a Gauss-Newton and steepest descent direction. LM algorithm computes a new direction using a following formula:

$$\Delta \mathbf{x} = -\left(\mathbf{H}_{\mathbf{x}} + \lambda \mathbf{I}\right)^{-1} \mathbf{g}_{\mathbf{x}}, \tag{16}$$

where: λ is an internal parameter, $\mathbf{g}_{\mathbf{x}} = \nabla \omega\left(\mathbf{x}\right)$ is a gradient of objective function ω with respect to parameters $\mathbf{x}$:

$$\mathbf{g}_{\mathbf{x}} = \frac{\partial \omega}{\partial \mathbf{x}} \tag{17}$$

and Hessian $\mathbf{H}_{\mathbf{x}} = \nabla^2 \omega\left(\mathbf{x}\right)$ is a second partial derivative of ω with respect to parameters $\mathbf{x}$:

$$\mathbf{H}_{\mathbf{x}} = \frac{\partial^2 \omega}{\partial \mathbf{x}^2}. \tag{18}$$

In the nonlinear least square approach, the gradient and Hessian can be computed based on Jacobian:

$$\mathbf{J}\left(\mathbf{x}\right) = \frac{\partial \mathbf{R}}{\partial \mathbf{x}}, \tag{19}$$

so the gradient and Hessian are defined, respectively:

$$\mathbf{g}\left(\mathbf{x}\right) = \mathbf{J}^T \mathbf{R}, \qquad \mathbf{H}\left(\mathbf{x}\right) \simeq \mathbf{J}^T \mathbf{J}. \tag{20}$$

Such approximation of the Hessian, which can be computed 'for free' once the Jacobian is available, represents a distinctive feature of least squares problems. This approximation is, however valid if residuals are small, meaning we are close to the solution, therefore some techniques may be required in order to precondition Hessian to be semi-positive defined (see e.g. [25]).

One of the main issues of the trust region approach, that to a large extent determines the success and the performance of this algorithm, is the decision strategy of how large the trusted region should be. Allowing it to be too large can make the algorithm facing the same problem as the classical Newton direction line search, when the minimizer of model function is quite far from the minimizer of the actual objective function. On the other hand using too small region the algorithm misses an opportunity to take a substantial step that could move it much closer to the solution.

Each k-th step in the trust region algorithm is obtained by solving the sub-problem defined by

$$\min_{\mathbf{d}_k} m_k\left(\mathbf{d}_k\right)=f\left(\mathbf{x}_k\right)+\mathbf{d}_k^T\nabla f\left(\mathbf{x}_k\right)+\frac{1}{2}\mathbf{d}_k^T\nabla^2 f\left(\mathbf{x}_k\right)\mathbf{d}_k, \qquad \left\|\mathbf{d}_k\right\|\leq\Delta_k, \tag{21}$$

where Δ_k is the trust region radius. By writing the unknown direction as a linear combination of Newton and steepest descend direction, the sub-problem will obtain the following form:

$$\begin{aligned}\min m_k\left(\mathbf{x}_k\right)= & \; f\left(\mathbf{x}_k\right)+\left[\alpha_1\mathbf{d}_k^{SD}+\alpha_2\mathbf{d}_k^{N}\right]^T\nabla f\left(\mathbf{x}_k\right)+\\ + & \; \tfrac{1}{2}\left[\alpha_1\mathbf{d}_k^{SD}+\alpha_2\mathbf{d}_k^{N}\right]^T\nabla^2 f\left(\mathbf{x}_k\right)\left[\alpha_1\mathbf{d}_k^{SD}+\alpha_2\mathbf{d}_k^{N}\right]\end{aligned} \tag{22}$$

under the constrains:

$$\left\|\alpha_1\mathbf{d}_k^{SD}+\alpha_2\mathbf{d}_k^{N}\right\|\leq\Delta_k. \tag{23}$$

The problem now becomes two dimensional and it is solved for the unknown coefficients α_1 and α_2. In order to find both alphas in (22) the set of nonlinear equations has to be solved using for example a Newton-Raphson techniques. Herein this approach is implemented into inverse procedure for the discrepancy minimization.

TRA, however, is a 'local' algorithm and if objective function is non-convex it may stuck in the local minimum. Therefore a regularization method or multi-start techniques can be beneficial. Here, very good results are obtained when the procedure is divided into two steps: first decomposition of the output structural response signal, e.g. strains, using DWT (see Fig. 3) for preliminary estimate of damage location and second the application of TRA on the limited search field (i.e. to the number of elements, where the wavelet disturbances is clearly evidenced). The advantage of this approach is relatively small number of iterations where damage details/sought parameters are properly specified (see Fig. 4).

Another possible technique which can be applied to minimize the non-convex functions are methods belonging to the global minimization search family. Here, Genetic Algorithm (GA) is programmed and employed for the damage detection problem. Some more details on GA and other evolutionary-based algorithms can be found in many textbooks and articles (e.g. [26], [27]). Unfortunately, the first approach to the damage detection problem with GA was unsuccessful (i.e. none of the sought parameters were found) when as

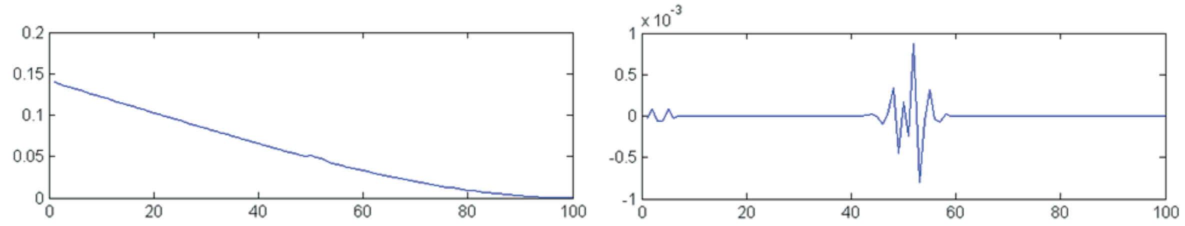

Figure 3: Structural response signal: a) expressed in strains , b) decomposed using DWT (detail 1).

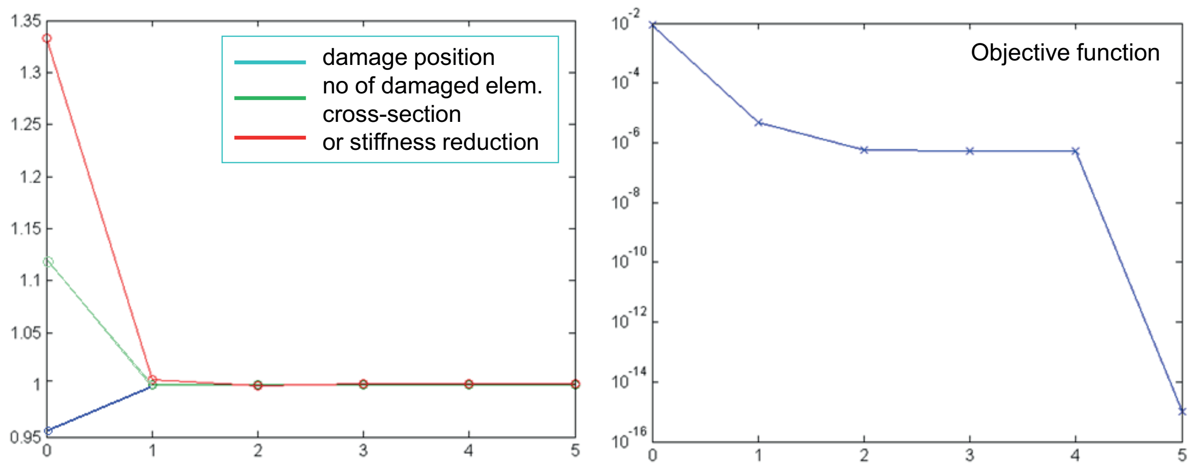

Figure 4: Trust region algorithm: a) identified damage details, b) objective function.

the output signal the direct structural dynamic response (e.g. expressed in strains) was used (see Fig. 5). Only the solution provided by application of GA on the output signal represented by wavelet coefficients appeared to be successful (see Fig. 6). All defect details, such as location, intensity, shape or number, were clearly identified within relatively small number of iterations.

5 CONCLUSIONS

The contribution of this work is a novel approach to Structural Health Monitoring (SHM) based on damage detection through wavelet transformation, numerical FE modeling and mathematical programming. The inverse analysis employed here uses two distinct minimization algorithms in order to select the most suitable technique of DWT application to SHM. The effectiveness of the method is studied by the way of an example of cantilever steel beam subjected to mechanical excitation. The eigenvibrations are considered. The examples proved that application either TRA or GA is very efficient in determining the details of damage such as location, severity, shape or number of defective elements. However, the prerequisite is that as the output the structural response signal (e.g. strains) represented in wavelet coefficients is taken into consideration. In the case of

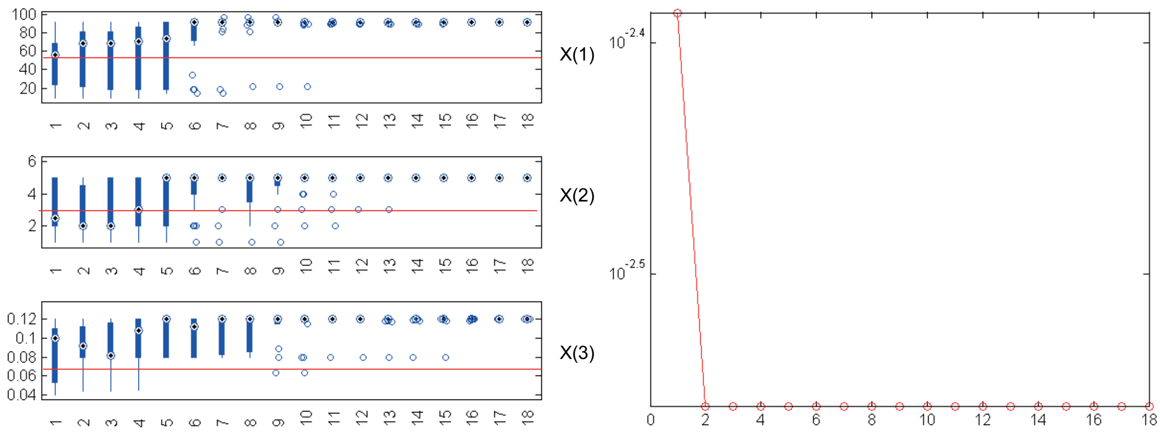

Figure 5: Genetic Algorithm: a) unidentified damage details, b) fitness function.

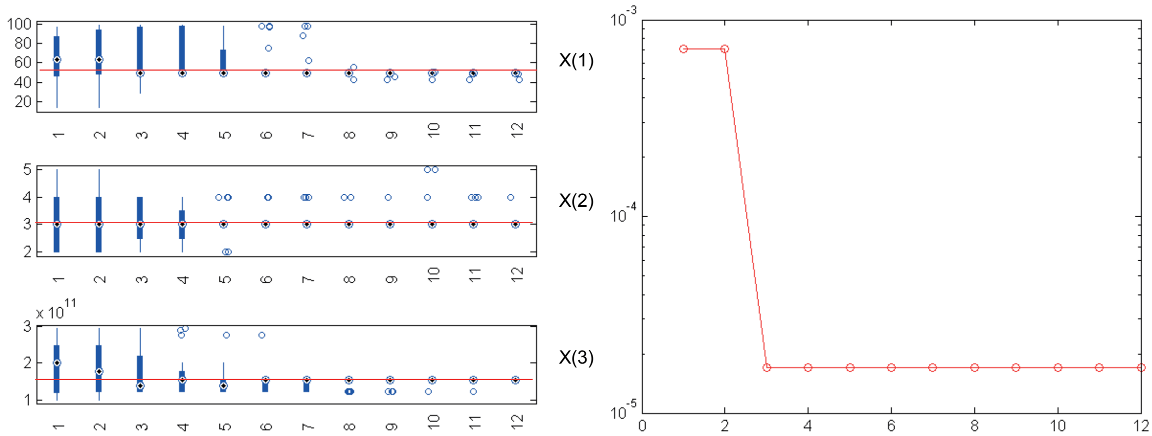

Figure 6: Genetic Algorithm: a) identified damage details, b) fitness function.

TRA application it allows to limit the search field to the number of elements, where the evident wavelet disturbances is visible, therefore the procedure is performed in a small number of iterations. In GA, when as the output signal the direct structural dynamic response is used, damage detection failed. Only when the output signal is expressed in wavelet coefficients, as mentioned, damage details are properly specified.

This preliminary work serves as a check of the usefulness of the proposed technique, and will be validated, in future, by a real experiment on structural elements.

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